Math 118 Winter 2015 Midterm Exam 1 Solutions

Feb. 6, 2015 10am - 10:50am. Total 20+2 Pts

NAME: ID#:

- There are five questions.
- Please write clearly and show enough work.

Question 1. (5 pts) *Calculate* $\int \frac{x+2}{x^3+x}\,\mathrm{d}x$.

Solution. Write

$$\frac{x+2}{x^3+x} = \frac{A}{x} + \frac{B\,x+C}{x^2+1}. \tag{1}$$

Multiply both sides by x and then let $x \to 0$ we have $A = 2$. This gives

$$\frac{B\,x+C}{x^2+1} = \frac{x+2}{x^3+x} - \frac{2}{x} = \frac{x-2\,x^2}{x^3+x} = \frac{-2\,x+1}{x^2+1} \tag{2}$$

so $B = -2, C = 1$. Thus

$$\int \frac{x+2}{x^3+x}\,\mathrm{d}x = \int \left[\frac{2}{x} + \frac{-2\,x+1}{x^2+1}\right] = \ln\left(\frac{x^2}{x^2+1}\right) + \arctan x + C. \tag{3}$$

Question 2. (5 pts) *Calculate* $\int \frac{\sin^3 x}{\cos^5 x}\,\mathrm{d}x$.

Solution. We have

$$\begin{aligned} \int \frac{\sin^3 x}{\cos^5 x}\,\mathrm{d}x &= \int \tan^3 x\,\mathrm{d}\tan x \\ &= \frac{1}{4}\tan^4 x + C. \end{aligned}$$

Question 3. (5 pts) *Calculate* $\int_1^e \frac{\ln x}{x^2}\,\mathrm{d}x$.

Solution. We have

$$\begin{aligned}\int_1^e \frac{\ln x}{x^2}\,\mathrm{d}x &= \int_1^e \ln x\,\mathrm{d}\Big(-\frac{1}{x}\Big)\\ &= -\frac{\ln x}{x}\Big|_1^e + \int_1^e \frac{1}{x^2}\,\mathrm{d}x\\ &= 1-2\,e^{-1}.\end{aligned}$$

Question 4. (5 pts) *Calculate* $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2+2\,x}}$.

Solution.

- Method 1. We have

$$\sqrt{x^2+2\,x} = x^{1/2}\,(x+2)^{1/2} = x\left(\frac{x+2}{x}\right)^{1/2}. \tag{4}$$

Set $t=\left(\frac{x+2}{x}\right)^{1/2}$. Then we have

$$x=\frac{2}{t^2-1}; \qquad \mathrm{d}x = -\frac{4\,t}{(t^2-1)^2}\,\mathrm{d}t. \tag{5}$$

Thus

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+2\,x}} = \int -\frac{2}{t^2-1}\,\mathrm{d}t = \ln\left|\frac{t+1}{t-1}\right| + C = \ln\left|x+1+\sqrt{x^2+2\,x}\right| + C. \tag{6}$$

- Method 2. We write $\sqrt{x^2+2\,x} = x-t$. This gives

$$x=\frac{t^2}{2\,(t+1)}, \quad \sqrt{x^2+2\,x} = \frac{-t^2-2\,t}{2\,(t+1)}, \quad \mathrm{d}x = \frac{t^2+2\,t}{2\,(t+1)^2}\,\mathrm{d}t \tag{7}$$

Thus

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+2\,x}} = \int -\frac{1}{t+1}\,\mathrm{d}t = -\ln|t+1| + C. \tag{8}$$

This simplifies to

$$\ln\left|x+1+\sqrt{x^2+2\,x}\right| + C. \tag{9}$$

- Method 3.

  First set $u=x+1$. We have

$$\int \frac{\mathrm{d}x}{\sqrt{x^2+2\,x}} = \int \frac{\mathrm{d}u}{\sqrt{u^2-1}}. \tag{10}$$

Now let $u = \cosh t := \frac{e^t + e^{-t}}{2}$ with $t > 0$. Then we have

$$\int \frac{\mathrm{d}u}{\sqrt{u^2 - 1}} = \int \mathrm{d}t = t + C = \ln\left|u + \sqrt{u^2 - 1}\right| + C. \tag{11}$$

So

$$\int \frac{\mathrm{d}x}{\sqrt{x^2 + 2\,x}} = \ln\left|x + 1 + \sqrt{x^2 + 2\,x}\right| + C. \tag{12}$$

Question 5. (Extra 2 pts) *Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ be differentiable with continuous non-vanishing $f'(x)$ and invertible with inverse function $g(x)$. Further assume that $f(x)$ is elementary. Prove: $\int f(x)\,\mathrm{d}x$ is elementary if and only if $\int g(x)\,\mathrm{d}x$ is elementary.*

Proof. Let $\int g(x) = G(x) + C$. Now we have

$$\begin{aligned} \int f(x)\,\mathrm{d}x &= x\,f(x) - \int x\,\mathrm{d}f(x) \\ &= x\,f(x) - \int g(f(x))\,\mathrm{d}f(x) \\ &\overset{u=f(x)}{=\!=} x\,f(x) - \int g(u)\,\mathrm{d}u \\ &= x\,f(x) - G(f(x)) + C. \end{aligned} \tag{13}$$

The conclusion follows. ☐