Math 118 Winter 2015 Lecture 15 (Jan. 29, 2015)

- Recall:
  - Integration by parts:
$$\int_a^b u(x)\, v'(x)\, \mathrm{d}x = u(b)\, v(b) - u(a)\, v(a) - \int_a^b u'(x)\, v(x)\, \mathrm{d}x. \tag{1}$$
    holds if
    i. u, v are continuous on $[a, b]$;
    ii. u, v are differentiable on (a, b);
    iii. u', v' are integrable on (a, b).
  - Change of variable:
$$\int_a^b f(u(t))\, u'(t)\, \mathrm{d}t = \int_{u(a)}^{u(b)} f(x)\, \mathrm{d}x. \tag{2}$$
    holds if
    i. f is continuous on $u([a, b])$;
    ii. u is continuous on $[a, b]$;
    iii. u is differentiable on (a, b);
    iv. u' is integrable on (a, b).
  - In practice: (1) and (2) hold as long as all the functions involved in the formulas are continuous on $[a, b]$.
- Simple examples

Example 1. Calculate
$$\int_0^1 x\, e^x\, \mathrm{d}x. \tag{3}$$

Solution. We set $v = e^x, u = x$. Then we have
$$\begin{aligned} \int_0^1 x\, e^x\, \mathrm{d}x &= \int_0^1 x\, \mathrm{d}e^x && (4)\\ &= x\, e^x|_0^1 - \int_0^1 e^x\, \mathrm{d}x && (5)\\ &= e - e^x|_{x=0}^{x=1} && (6)\\ &= 1. && (7) \end{aligned}$$

Example 2. Calculate
$$\int_0^{\pi/4} \frac{\mathrm{d}x}{\cos x}. \tag{8}$$

Solution. Set $y = \sin x$. We see that $[0, \pi/4]$ is mapped to $[\sin 0, \sin(\pi/4)] = \left[0, \frac{\sqrt{2}}{2}\right]$.

Now calculate

$$\begin{aligned}
\int_0^{\pi/4} \frac{\mathrm{d}x}{\cos x} &= \int_0^{\pi/4} \frac{\cos x\,\mathrm{d}x}{\cos^2 x} \\
&= \int_0^{\pi/4} \frac{\cos x\,\mathrm{d}x}{1-\sin^2 x} \\
&= \int_0^{\sqrt{2}/2} \frac{\mathrm{d}y}{1-y^2} \\
&= -\frac{1}{2}\int_0^{\sqrt{2}/2} \left[\frac{\mathrm{d}y}{y-1} - \frac{\mathrm{d}y}{y+1}\right] \\
&= -\frac{1}{2}\left[\ln|y-1| - \ln|y+1|\right]_{y=0}^{y=\sqrt{2}/2} \\
&= \frac{1}{2}\ln\left(\frac{y+1}{1-y}\right)_{y=0}^{y=\sqrt{2}/2} \\
&= \frac{1}{2}\ln\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right). \qquad (9)
\end{aligned}$$

Exercise 1. Prove the following:

a) Let f be even and integrable on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x. \qquad (10)$$

b) Let f be odd and integrable on $[-a, a]$, then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 0. \qquad (11)$$

Example 3. Calculate $I := \int_0^1 x^2\sqrt{1-x^2}\,\mathrm{d}x$.

Solution. Set $x = \sin t$. Thus we have

$$\begin{aligned}
\int_0^1 x^2\sqrt{1-x^2}\,\mathrm{d}x &= \int_0^{\pi/2} \sin^2 t\cos^2 t\,\mathrm{d}t \\
&= \frac{1}{4}\int_0^{\pi/2} \sin^2 2\,t\,\mathrm{d}t \\
&= \frac{1}{8}\int_0^{\pi/2} [1-\cos 4\,t]\,\mathrm{d}t \\
&= \frac{\pi}{16}. \qquad (12)
\end{aligned}$$

- More involved examples.

Example 4. Calculate $I_n := \int_0^{\pi/2} \sin^n x\,\mathrm{d}x$.

Solution. Through integration by parts, we have

$$I_n = (n-1)\,I_{n-2} - (n-1)\,I_n \Longrightarrow I_n = \frac{n-1}{n}\,I_{n-2}. \qquad (13)$$

As

$$I_0 = \frac{\pi}{2}, \qquad I_1 = 1, \qquad (14)$$

we have

$$I_n = \begin{cases} \dfrac{(2\,k-1)\,(2\,k-3)\cdots 3\cdot 1}{2\,k\,(2\,k-2)\cdots 4\cdot 2}\,\dfrac{\pi}{2} & n=2\,k \\ \dfrac{2\,k\,(2\,k-2)\cdots 4\cdot 2}{(2\,k+1)\,(2\,k-1)\cdots 3\cdot 1} & n=2\,k+1 \end{cases}. \tag{15}$$

Exercise 2. Calculate $J_n := \int_0^{\pi/2} \cos^n x \, \mathrm{d}x$.

Exercise 3. Prove the Wallis formula:

$$\frac{\pi}{2} = \lim_{n\to\infty} \left[\frac{2\,n\,(2\,n-2)\cdots 4\cdot 2}{(2\,n-1)\,(2\,n-3)\cdots 3\cdot 1} \right]^2 \frac{1}{2\,n+1}. \tag{16}$$

(Hint:[1])

Note. In practice we often deal with definite integrals with singularities, such as $a=-\infty$, $b=\infty$, or f is unbounded on $[a,b]$. Such integrals are not covered by the theory of Riemann integrals but by a simple generalization of it, the theory of improper integrals. We will deal with this theory in a couple of weeks. For now, let's just accept that FTC1 as well as (1) and (2) still hold for such integrals, as long as everything are continuous and the integrals makes sense.

Example 5. Let $n\in\mathbb{N}$. Calculate $\int_0^\infty x^n\, e^{-x}\, \mathrm{d}x$.

Solution. Denote this integral by I_n. Then we have

$$\begin{aligned} I_n &= -\int_0^\infty x^n\, \mathrm{d}e^{-x} \\ &= -x^n\, e^{-x}|_0^\infty + n \int_0^\infty x^{n-1}\, e^{-x}\, \mathrm{d}x \\ &= n\, I_{n-1}. \end{aligned} \tag{17}$$

Exercise 4. Why do all the boundary terms vanish?

From this we easily deduce:

$$\int_0^\infty x^n\, e^{-x}\, \mathrm{d}x = n!. \tag{18}$$

Exercise 5. Calculate $\int_0^\infty e^{-x} \sin x \, \mathrm{d}x$, $\int_0^\infty e^{-x} \cos x \, \mathrm{d}x$.

Definition 6. *Let $t>0$. We define the "Gamma function" $\Gamma(t)$ as*

$$\Gamma(t) := \int_0^\infty x^{t-1}\, e^{-x}\, \mathrm{d}x. \tag{19}$$

Exercise 6. Prove that $\Gamma(t+1) = t\,\Gamma(t)$.

1. Start from proving $\int_0^{\pi/2} \sin^{2n+1} x \ \mathrm{d}x < \int_0^{\pi/2} \sin^{2n} x \ \mathrm{d}x < \int_0^{\pi/2} \sin^{2n-1} x \ \mathrm{d}x$. Then apply Squeeze Theorem on $1 < \frac{I_{2n}}{I_{2n+1}} < \frac{I_{2n-1}}{I_{2n+1}}$.