Math 118 Winter 2015 Lecture 12 (Jan. 23, 2015)

- Integrals of the form $\int R\left(x,\, \left(\frac{a\,x+b}{c\,x+d}\right)^{n_1/m_1},\, \ldots,\, \left(\frac{a\,x+b}{c\,x+d}\right)^{n_k/m_k}\right) \mathrm{d}x$ where $R(x,\, y)$ is rational.

  Note. We say a multi-variable function $R(x_1,\, \ldots,\, x_k)$ is rational if there are polynomials $P(x_1, \ldots, x_k)$ and $Q(x_1, \ldots, x_k)$ such that $R = P/Q$. We say $P(x_1, \ldots, x_k)$ is a polynomial if it is a sum of finitely many terms of the form $a_{i_1 \ldots i_k} x_1^{i_1} \cdots x_k^{i_k}$.

  Lemma 1. *Let $m_1,\, \ldots,\, m_k,\, n_1,\, \ldots,\, n_k \in \mathbb{N}$ and $R(x_1,\, \ldots,\, x_{k+1})$ be rational. Then there is a rational function $R_1(x, y)$ and $m \in \mathbb{N}$ such that*

$$R\left(x, \left(\frac{a\,x+b}{c\,x+d}\right)^{n_1/m_1}, \ldots, \left(\frac{a\,x+b}{c\,x+d}\right)^{n_k/m_k}\right) = R_1\left(x, \left(\frac{a\,x+b}{c\,x+d}\right)^{1/m}\right). \tag{1}$$

  In fact we can take m to be any number dividable by $m_1, \ldots, m_k$.

  Proof. Exercise. □

  Proposition 2. *Let $R(u,\, v)$ be rational. Then $\int R\left(x,\, \left(\frac{a\,x+b}{c\,x+d}\right)^{1/m}\right) \mathrm{d}x$ can always be integrated through $t = \left(\frac{a\,x+b}{c\,x+d}\right)^{1/m}$, if $a\,d - b\,c \neq 0$.*

  Proof. Exercise. □

  Exercise 1. What if $a\,d - b\,c = 0$?

  Exercise 2. Does the above theory cover $\int \frac{\sqrt{x+1}}{x + \sqrt{x+3}} \mathrm{d}x$?

  Example 3. $\int \frac{1-\sqrt{x+1}}{1 + \sqrt[3]{x+1}} \mathrm{d}x$.

  Solution. Let $(x+1)^{1/6} = t$. Then we have

$$\begin{aligned}
\int \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}} \mathrm{d}x &= 6 \int \frac{t^5 - t^8}{1+t^2} \mathrm{d}t \\
&= 6 \int \left[-t^6 + t^4 + t^3 - t^2 - t + 1 + \frac{t-1}{1+t^2}\right] \mathrm{d}t \\
&= -\frac{6}{7} t^7 + \frac{6}{5} t^5 + \frac{3}{2} t^4 - 2\,t^3 - 3\,t^2 + 6\,t + 3\ln(1+t^2) - 6\arctan t + C \\
&= -\frac{6}{7}(x+1)^{7/6} + \frac{6}{5}(x+1)^{5/6} + \frac{3}{2}(x+1)^{2/3} \\
&\quad -2\,(x+1)^{1/2} - 3\,(x+1)^{1/3} + 6\,(x+1)^{1/6} \\
&\quad +3\ln\left|1 + (1+x)^{1/3}\right| - 6\arctan(1+x)^{1/6} + C.
\end{aligned} \tag{2}$$

  Exercise 3. Calculate the following integrals:

$$\int \frac{x}{1+\sqrt[3]{x^2}} \mathrm{d}x; \qquad \int \frac{\sqrt{x}+x}{\sqrt[4]{x}+1} \mathrm{d}x; \qquad \int \frac{\sqrt{x+5} + \sqrt[3]{x+5}}{x+1} \mathrm{d}x. \tag{3}$$

- Binomial differentials.

  Proposition 4. *Let $r, s \in \mathbb{Q}$. $\int x^r (1-x)^s \, \mathrm{d}x$ can be integrated in any one of the following three situations: $r \in \mathbb{Z}, s \in \mathbb{Z}, r+s \in \mathbb{Z}$.*

Proof. The conclusions follow immediately from Proposition 2. For example, when $r+s\in\mathbb{Z}$ we have $x^r\,(1-x)^s=x^{r+s}\left(\frac{1-x}{x}\right)^s$. ☐

Note. $x^m\,(a+b\,x^n)^p\,\mathrm{d}x$, with $a,b\in\mathbb{R}$, $m,n,p\in\mathbb{Q}$, is called a "differential binomial".

THEOREM 5. *Let $a,b\in\mathbb{R},m,n,p\in\mathbb{Q}$. $\int x^m\,(a+b\,x^n)^p\,\mathrm{d}x$ can be integrated in any one of the following three situations:*

- $p\in\mathbb{Z}$;
- $\frac{m+1}{n}\in\mathbb{Z}$;
- $\frac{m+1}{n}+p\in\mathbb{Z}$.

Proof. Exercise. ☐

Example 6. Calculate $\int\sqrt{x^2+\frac{1}{x^2}}\,\mathrm{d}x$.

Solution. We have

$$\int\sqrt{x^2+\frac{1}{x^2}}\,\mathrm{d}x=\int x^{-1}\,(1+x^4)^{1/2}\,\mathrm{d}x. \tag{4}$$

We see that it is integral of a differential binomial with $a=b=1,m=-1,n=4,p=1/2$. We see that $\frac{m+1}{n}\in\mathbb{Z}$ therefore we set $t=\sqrt{1+x^4}$. This gives $x=(t^2-1)^{1/4}$ and transform the integral to

$$\frac{1}{2}\int\frac{t^2}{t^2-1}\,\mathrm{d}t=\frac{t}{2}-\frac{1}{4}\ln\left|\frac{t+1}{t-1}\right|+C. \tag{5}$$

Substituting back $t=\sqrt{1+x^4}$ we reach

$$\int\sqrt{x^2+\frac{1}{x^2}}\,\mathrm{d}x=\frac{1}{4}\ln\left|\frac{1-\sqrt{1+x^4}}{1+\sqrt{1+x^4}}\right|+\frac{1}{2}\sqrt{1+x^4}+C. \tag{6}$$

Remark 7. In 1853 P. L. Chebyshev proved that besides the three cases $r\in\mathbb{Z},s\in\mathbb{Z},r+s\in\mathbb{Z}$, $\int x^r\,(1-x)^s\,\mathrm{d}x$ cannot be integrated – note that in the context of indefinite integration, this means the anti-derivative is not elementary, that is cannot be written down using x, e, ln and trig/inverse trig functions – thus the three cases listed in Theorem 5 are the only ones that can be integrated.[1] The proof requires complex analysis but we will try to explain the main ideas in the next lecture.

> **Exercise 4.** Consider $\int\sqrt{x^k+x^{-k}}\,\mathrm{d}x$ where $k=1,2,3,4,5$. For which k can we carry out the integration?

- Integrals of the form $\int R\left(x,\sqrt{a\,x^2+b\,x+c}\right)\mathrm{d}x$ where $R(x,y)$ is rational.

  The basic idea is the following: We try to find a rational function $R_1(t)$ such that when $x=R_1(t)$, $y=\sqrt{a\,x^2+b\,x+c}=R_2(t)$ is also rational.

1. I saw this on p.166 of *Handbook of Conformal Mapping with Computer-Aided Visualization* by V. I. Ivanov and M. K. Trubetskov: The integral can also be done when one of r,s is a natural number and the other is irrational.

Exercise 5. Show that once this is done the integral can be evaluated.

First notice that, if $a\,x^2+b\,x+c$ has real roots, then we have if $a>0$,

$$\sqrt{a\,x^2+b\,x+c}=\sqrt{a}\,\sqrt{(x-x_1)\,(x-x_2)}=\sqrt{a}\,(x-x_1)\,\sqrt{\frac{x-x_2}{x-x_1}} \tag{7}$$

or if $a<0$

$$\sqrt{a\,x^2+b\,x+c}=\sqrt{-a}\,\sqrt{(x-x_1)\,(x_2-x)}=\sqrt{-a}\,(x-x_1)\,\sqrt{\frac{x_2-x}{x-x_1}} \tag{8}$$

both belongs to the case discussed in the beginning of this lecture.

If $a\,x^2+b\,x+c$ has no real roots and $a<0$, then $a\,x^2+b\,x+c<0$ for all $x\in\mathbb{R}$ and it is meaningless to discuss $\int R(x,\sqrt{a\,x^2+b\,x+c})\,\mathrm{d}x$.

Thus the only nontrivial case is $a>0$ and $a\,x^2+b\,x+c>0$ for all $x\in\mathbb{R}$. In this case clearly it suffices to consider the case $a=1$.

Set $t=\sqrt{x^2+b\,x+c}-x$.

Exercise 6. Prove that t and x are one-to-one. What is the range of t as a function of x?

This gives

$$t^2+2\,x\,t+x^2=x^2+b\,x+c\Longrightarrow x=\frac{t^2-c}{b-2\,t}\Longrightarrow\sqrt{x^2+b\,x+c}=x+t=\frac{b\,t-c-t^2}{b-2\,t}. \tag{9}$$

We see that $\int R(x,\sqrt{x^2+b\,x+c})=\int R_1(t)\,\mathrm{d}t$ where R_1 is rational.

Exercise 7. Would setting $t=\sqrt{x^2+b\,x+c}+x$ work? Justify.

Example 8. $\displaystyle\int\frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}}$.

Solution. We see that $x^2-x+1>0$ for all $x\in\mathbb{R}$. So setting $t=\sqrt{x^2-x+1}-x$ obtains

$$x=\frac{1-t^2}{2\,t+1}\Longrightarrow\mathrm{d}x=\frac{-2\,t^2-2\,t-2}{(2\,t+1)^2}. \tag{10}$$

. Therefore

$$\begin{aligned}\int\frac{\mathrm{d}x}{x+\sqrt{x^2-x+1}} &= \int\frac{1}{t+2\,\frac{1-t^2}{2\,t+1}}\,\frac{-2\,t^2-2\,t-2}{(2\,t+1)^2}\,\mathrm{d}t\\ &= \int\frac{-2\,t^2-2\,t-2}{(t+2)\,(2\,t+1)}\,\mathrm{d}t\\ &= -\int\mathrm{d}t+\frac{3}{2}\left[\int\frac{4/3}{t+2}\,\mathrm{d}t+\int\frac{-1/3}{t+1/2}\,\mathrm{d}t\right]\\ &= -t+2\ln|t+2|-\frac{1}{2}\ln\left|t+\frac{1}{2}\right|+C.\end{aligned} \tag{11}$$

Thus the final answer is

$$x-\sqrt{x^2-x+1}+2\ln\left|\sqrt{x^2-x+1}-x+2\right|-\frac{1}{2}\ln\left|\sqrt{x^2-x+1}-x+\frac{1}{2}\right|+C. \tag{12}$$

Exercise 8. Solve the same example through setting $t=\sqrt{x^2-x+1}+x$.[2]

2. It's a bit surprising that this leads to a more complicated integral!