Math 118 Winter 2015 Lecture 10 (Jan. 21, 2015)

- The method of partial fractions: To evaluate $\int \frac{P(x)}{Q(x)}\,\mathrm{d}x$ where P, Q are polynomials,

  1. Check $\deg P < \deg Q$. If not perform $P = Q\,P_0 + R$ with P_0, R polynomials and $\deg R < \deg Q$ and write
  $$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x = \int P_0(x)\,\mathrm{d}x + \int \frac{R(x)}{Q(x)}\,\mathrm{d}x. \tag{1}$$
  2. Factorize $Q(x)$:
  $$Q(x) = (x-a_1)^{k_1}\cdots(x-a_l)^{k_l}(x^2+p_1\,x+q_1)^{m_1}\cdots(x^2+p_r\,x+q_r)^{m_r} \tag{2}$$
  where each $p_i^2 < 4\,q_i$ for all $i = 1, 2, \ldots, r$.
  3. Calculate the partial fraction resolution of $\frac{R}{Q}$:
  $$\begin{aligned}\frac{R(x)}{Q(x)} &= \frac{A_{11}}{x-a_1} + \cdots + \frac{A_{1k_1}}{(x-a_1)^{k_1}} \\ &\quad + \cdots + \frac{A_{l1}}{x-a_l} + \cdots + \frac{A_{lk_l}}{(x-a_l)^{k_l}} \\ &\quad + \frac{B_{11}x + C_{11}}{x^2+p_1\,x+q_1} + \cdots + \frac{B_{1m_1}\,x + C_{1m_1}}{(x^2+p_1\,x+q_1)^{m_1}} \\ &\quad + \cdots + \frac{B_{r1}\,x + C_{r1}}{x^2+p_r\,x+q_r} + \cdots + \frac{B_{rm_r}\,x + C_{rm_r}}{(x^2+p_r\,x+q_r)^{m_r}}.\end{aligned} \tag{3}$$
  Note that the number of coefficients to determine is the same as $\deg Q$.
  4. Integrate:
  $$\begin{aligned}\int \frac{P(x)}{Q(x)}\,\mathrm{d}x &= \int P_0(x)\,\mathrm{d}x + \int \frac{A_{11}}{x-a_1}\,\mathrm{d}x + \cdots + \int \frac{A_{1k_1}}{(x-a_1)^{k_1}}\,\mathrm{d}x \\ &\quad + \cdots + \int \frac{B_{r1}\,x + C_{r1}}{x^2+p_r\,x+q_r}\,\mathrm{d}x + \cdots + \int \frac{B_{rm_r}\,x + C_{rm_r}}{(x^2+p_r\,x+q_r)^{m_r}}\,\mathrm{d}x.\end{aligned} \tag{4}$$

  Exercise 1. Let $\deg P \geqslant \deg Q$. Assume that we forgot to do Step 1. What could go wrong?

Remark 1. Step 1 is trivial. Steps 3 and 4 may be tedious but can always be done with enough time. On the other hand, the factorization of $Q(x)$ depends on our ability to find all solutions to the equation $Q(x) = 0$. This is known to be in general not possible.

- Hermite's method.
  - From (3) it is clear that
  $$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x = F_1(x) + F_2(x) + F_3(x) \tag{5}$$
  where $F_1(x)$ is rational, $F_2(x)$ is of the form $\sum \frac{A}{\ln|x-a|}$ while F_3 is of the form $\sum \ln(x^2 + p\,x + q) + \sum \arctan(r\,x + s)$.[1]

    Problem 1. Prove that $f(x) := \sum_{i=1}^{k} A_i \ln|x - a_i|$ is not rational. That is there is no polynomials $P(x), Q(x)$ such that $f(x) = \frac{P(x)}{Q(x)}$ for all x, unless all $A_i = 0$. (Hint:[2])

1. If we allow complex numbers, arctan could be reduced to ln and $x^2 + p\,x + q$ can always be factorized. Then we have
$$\int \frac{P(x)}{Q(x)}\,\mathrm{d}x = \text{Rational function} + \sum \frac{A}{\ln|x-a|}. \tag{6}$$

Problem 2. For those who know partial derivatives: Let $Q(x) = (x - a_1)^{k_1} \cdots (x - a_l)^{k_l}$ and $\deg P < \deg Q$. Prove

$$\int \frac{P(x)}{Q(x)} = \frac{1}{(k_1 - 1)! \cdots (k_l - 1)!} \frac{\partial^{\sum k_i - l}}{\partial a_1^{k_1 - 1} \cdots \partial a_l^{k_l - 1}} \left[\sum_{i=1}^{l} \frac{P(a_i)}{Q'(a_i)} \ln|x - a_i| \right]. \tag{7}$$

Check:

$$\int \frac{\mathrm{d}x}{[(x-a)(x-b)]^2} = \frac{\partial^2}{\partial a \partial b} \left\{ \frac{1}{a-b} \ln\left| \frac{x-a}{x-b} \right| \right\}. \tag{8}$$

(Hint:[3])

- Charles Hermite (1822 – 1901) proposed in 1872 a method that calculates $F_1(x)$ without complete factorization of $Q(x)$.
- Square-free factorization.

  Let $Q(x)$ be a polynomial of degree n. Its "square-free factorization" is the factorization

$$Q = Q_1 Q_2^2 Q_3^3 \cdots Q_n^n \tag{10}$$

where each Q_i is "square-free", there is

$$\forall \text{polynomial } P \text{ not a constant}, \qquad P | Q_i \Longrightarrow P^2 \nmid Q_i, \tag{11}$$

and furthermore for every $i \neq j$, Q_i, Q_j are relatively prime.

Exercise 2. Prove that for every $i \neq j$, Q_i^i and Q_j^j are relatively prime.

Exercise 3. Can you define "square-free" factorization for a natural number?

Example 2. Let $Q = (x+1)(x-1)^2(x^2+1)$. Then we have $Q_1 = (x+1)(x^2+1)$, $Q_2 = (x-1)$, and $Q_3 = Q_4 = Q_5 = 1$.

LEMMA 3. *Let P be a polynomial with $\deg P < \deg Q$. We have the partial fraction resolution:*

$$\frac{P}{Q} = \frac{P_1}{Q_1} + \frac{P_2}{Q_2^2} + \cdots + \frac{P_n}{Q_n^n} \tag{12}$$

where $\deg P_i < i \deg Q_i$ for all $i = 1, 2, \ldots, n$.

The proof is by induction and we omit it.

Thus

$$\int \frac{P}{Q} \mathrm{d}x = \int \frac{P_1}{Q_1} \mathrm{d}x + \cdots + \int \frac{P_n}{Q_n^n} \mathrm{d}x \tag{13}$$

and all we need to do is be able to integrate rational functions of the form $\frac{P}{Q^k}$ where Q is square-free.

- The rational part of $\int \frac{P}{Q^k} \mathrm{d}x$.

LEMMA 4. *$Q(x)$ is square-free if and only if Q and Q' are relatively prime.*

Proof. Exercise. □

2. Consider $f'(x)$.
3. Prove

$$\frac{P}{Q} = \frac{1}{(k_1 - 1)! \cdots (k_l - 1)!} \frac{\partial^{\sum k_i - l}}{\partial a_1^{k_1 - 1} \cdots \partial a_l^{k_l - 1}} \left(\frac{P}{Q_0} \right) \tag{9}$$

where $Q_0(x) = (x - a_1) \cdots (x - a_l)$.

COROLLARY 5. *Let P be arbitrary. There are polynomials C, D such that*

$$C\,Q + D\,Q' = P. \tag{14}$$

Furthermore we can take $\deg C < \deg Q'$, $\deg D < \deg Q$.

Proof. Omitted. □

Thus we have

$$\begin{aligned}\int \frac{P}{Q^k}\,\mathrm{d}x &= \int \frac{C\,Q + D\,Q'}{Q^k}\,\mathrm{d}x \\ &= \int \frac{C + \frac{1}{k-1}D'}{Q^{k-1}}\,\mathrm{d}x - \frac{D}{(k-1)\,Q^{k-1}}.\end{aligned} \tag{15}$$

We notice that apply the same procedure to the first integral would first reduce the power to Q^{k-2}. This can be $k-1$ times untile we have

$$\int \frac{P}{Q^k}\,\mathrm{d}x = \text{rational function} + \int \frac{R}{Q}\,\mathrm{d}x. \tag{16}$$

Exercise 4. Prove that $\int \frac{R}{Q}\,\mathrm{d}x$ is not rational.

Example 6. Calculate $\int \frac{4\,x^9 + 21\,x^6 + 2\,x^3 - 3\,x^2 - 3}{(x^7 - x + 1)^2}\,\mathrm{d}x$.[4]

Solution. We have $Q(x) = x^7 - x + 1$ which gives $Q'(x) = 7\,x^6 - 1$. We easily check that Q, Q' are relatively prime and therefore Q is square-free and Hermite's method applies.

To find C, D such that

$$C(x)\,(x^7 - x + 1) + D(x)\,(7\,x^6 - 1) = 4\,x^9 + 21\,x^6 + 2\,x^3 - 3\,x^2 - 3 \tag{17}$$

we write $C(x) = c_5\,x^5 + \cdots + c_0$ and $D(x) = d_6\,x^6 + \cdots + d_0$ and solve the 13×13 linear system to obtain $C(x) = -3\,x^2, D(x) = x^3 + 3$. Now as $C + D' = 0$, (15) reduces to

$$\int \frac{P}{Q^2} = -\frac{D}{Q} = -\frac{x^3 + 3}{x^7 - x + 1}. \tag{18}$$

LEMMA 7. *Let Q be square free.* $\int \frac{P}{Q^2}\,\mathrm{d}x$ *is rational if and only if* $P\,Q'' - P'\,Q'$ *is divisible by Q.*

Proof. We first prove "if". Let C, D be such that $C\,Q + D\,Q' = P$. Then

$$P\,Q'' - P'\,Q' = C\,Q\,Q'' - C'\,Q\,Q' - (C + D')\,(Q')^2. \tag{19}$$

The assumption now becomes $Q|\,(C + D')\,(Q')^2$. As $(Q, Q') = 1$, there must hold $Q|$ $(C + D')$, that is there is a polynomial H such that $\frac{C + D'}{Q} = H$. Now (15) gives

$$\int \frac{P}{Q^2}\,\mathrm{d}x = -\frac{D}{Q} + \int H \tag{20}$$

which is rational.

4. Taken from G. H. Hardy, *The Integration of Functions of a Single Variable*, 2ed, Cambridge, 1928.

Next we prove "only if". By (15) $\int \frac{P}{Q^2}\,\mathrm{d}x$ is rational if and only if $Q|\,(C+D')$. But this immediately gives $Q|\,(P Q''-P'\,Q')$. □

Problem 3. Let Q be square-free. Prove that $\int \frac{P}{Q^3}\,\mathrm{d}x$ is rational if and only if $P\,(3\,(Q'')^2-Q'\,Q''')-3\,P'\,Q'\,Q''+P''\,(Q')^2$ is divisible by Q.

- The calculation of square-free factoriztion (without actual factorization to irreducible factors!)

LEMMA 8. *Let $Q(x)=(x-a_1)^{k_1}\cdots(x-a_l)^{k_l}$ and $Q_0(x)=(x-a_1)\,(x-a_2)\cdots(x-a_l)$. Then*

$$Q_0(x)=\frac{Q(x)}{\gcd(Q,Q')}. \tag{21}$$

Proof. Exercise. □

Using Lemma 8 repeatedly, we can carry out the square-free factorization for any polynomial.

Example 9. $Q(x)=x^8+6\,x^6+12\,x^4+x^2$.[5]

Solution. We could have first factorize $Q=x^2\,(x^6+6\,x^4+12\,x^2+1)$ and factorize $x^6+6\,x^4+12\,x^2+1$. However we do not do this for the illustration of the method.

We have $Q'(x)=8\,x^7+36\,x^5+48\,x^3+16\,x$ and it can be calculated

$$\gcd(Q,Q')=x^5+4\,x^3+4\,x. \tag{22}$$

Thus $Q_0(x)=x^3+2\,x$. If we pretend that we cannot factorize Q_0, we see that there are three (complex) roots a_1,a_2,a_3 to $Q(x)$ and $Q_0(x)=(x-a_1)\,(x-a_2)\,(x-a_3)$. To figure out the power of each factor, we calculate

$$\gcd(Q_0,\gcd(Q,Q'))=x^3+2\,x. \tag{23}$$

Thus we have $Q(x)=Q_0^2\,(x^2+2)=\left(\frac{Q_0}{x^2+2}\right)^2(x^2+2)^3=x^2\,(x^2+2)^3$. As both x, x^2+2 are square-free, we see that we have obtained the square-free factorization of Q.

Exercise 5. Explain why $\frac{Q_0}{x^2+2}$ must be a polynomial.

Problem 4. Calculate $\int \frac{x^7-24\,x^4-4\,x^2+8\,x-8}{x^8+6\,x^6+12\,x^4+x^2}\,\mathrm{d}x$ through 1) partial fraction, 2) Hermite's method.

- Partial fraction for counting.

Example 10. [6]Let $a_1,\ldots,a_k$ be distinct natural numbers such that the only common factor is 1. Let $C(n)$ be the number of ways to write n as a sum of numbers from $A=\{a_1,\ldots,a_k\}$, a number can appear more than one times, and the order of the addition does not matter. One example of such problem is "How many ways are there to break 100 dollars into coins?"

We claim that $C(n)$ is the coefficient of z^n in the expansion of

$$(1+z^{a_1}+z^{2a_1}+\cdots)\cdots(1+z^{a_k}+z^{2a_k}+\cdots). \tag{24}$$

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5. Taken from M. Bronstein, *Symbolic Integration: Transcendental Functions*, 2ed, Springer, 2005.

6. Taken from Donald J. Newman, *Analytic Number Theory*, GTM177, Springer, 1998.

Thus recalling the Taylor expansion formula we formally have

$$\sum C(n)\, z^n = \frac{1}{(1 - z^{a_1})\cdots(1 - z^{a_k})}. \tag{25}$$

Exercise 6. Explain why the argument (24) $\Longrightarrow$ (25) is only formal. What do we need to prove here?

We claim that the only common factor in $1 - z^{a_1}, \ldots, 1 - z^{a_k}$ is $1 - z$.[7] Then we know in the partial fraction reduction

$$\frac{1}{(1 - z^{a_1})\cdots(1 - z^{a_k})} = \frac{c}{(1 - z)^k} + \text{other terms}. \tag{26}$$

As each term in "other terms" is of the form $\frac{c'}{(\alpha - z)^{k'}}$ with $k' < k$, we see that the leading term in $C(n)$ comes from $\frac{c}{(1 - z)^k}$. To figure out c, we multiply both sides by $(1 - z)^k$ and take limit $z \longrightarrow 1$ to obtain

$$c = \frac{1}{a_1 \cdots a_k}. \tag{27}$$

Exercise 7. Prove (27).

Problem 5. Prove that $C(n) \sim \frac{n^{k-1}}{a_1 \cdots a_k\, (k-1)!}$ in the sense that the ratio between the two $\longrightarrow 1$ as $n \longrightarrow \infty$.

7. To see this recall that the roots of $1 - z^a$ are $e^{i\theta_m}$, $m = 0, 1, 2, \ldots, a - 1$, with $\theta_m := \frac{m}{a}\, 2\, \pi\, i$.