Math 118 Winter 2015 Lecture 7 (Jan. 15, 2015)

- Integration by parts.

$$\begin{aligned}\int f(x)\,\mathrm{d}x &= \int u(x)\,v'(x)\,\mathrm{d}x\\ &= \int u\,\mathrm{d}v\\ &= u(x)\,v(x)-\int v\,\mathrm{d}u\\ &= u(x)\,v(x)-\int v(x)\,u'(x)\mathrm{d}x. \qquad (1)\end{aligned}$$

Hope: $v(x)\,u'(x)$ is easier to integrate than $f(x)$.

Example 1. Calculate $\int (3\,x^2-7\,x+1)\,e^x\,\mathrm{d}x$.

Solution. We have

$$\begin{aligned}\int (3\,x^2-7\,x+1)\,e^x\,\mathrm{d}x &= \int (3\,x^2-7\,x+1)\,\mathrm{d}e^x\\ &= (3\,x^2-7\,x+1)\,e^x-\int e^x\,\mathrm{d}(3\,x^2-7\,x+1)\\ &= (3\,x^2-7\,x+1)\,e^x-\int (6\,x-7)\,e^x\,\mathrm{d}x\\ &= (3\,x^2-7\,x+1)\,e^x-\int (6\,x-7)\,\mathrm{d}e^x\\ &= (3\,x^2-7\,x+1)\,e^x-(6\,x-7)\,e^x+\int e^x\,\mathrm{d}(6\,x-7)\\ &= (3\,x^2-7\,x+1)\,e^x-(6\,x-7)\,e^x+6\int e^x\,\mathrm{d}x\\ &= (3\,x^2-7\,x+1)\,e^x-(6\,x-7)\,e^x+6\,e^x+C\\ &= (3\,x^2-13\,x+14)\,e^x+C. \qquad (2)\end{aligned}$$

- Examples

Example 2. Calculate $\int e^x\cos x\,\mathrm{d}x$.

Solution. We have

$$\begin{aligned}\int e^x\cos x\,\mathrm{d}x &= \int \cos x\,\mathrm{d}e^x\\ &= e^x\cos x-\int e^x\,\mathrm{d}\cos x\\ &= e^x\cos x+\int e^x\sin x\,\mathrm{d}x\\ &= e^x\cos x+\int \sin x\,\mathrm{d}e^x\\ &= e^x\cos x+e^x\sin x-\int e^x\cos x\,\mathrm{d}x. \qquad (3)\end{aligned}$$

If we denote $I:=\int e^x\cos x\,\mathrm{d}x$ we have $I=e^x\,(\cos x+\sin x)-I\Longrightarrow I=\frac{1}{2}\,e^x\,(\cos x+\sin x)$. Therefore

$$\int e^x\cos x\,\mathrm{d}x=\frac{1}{2}\,e^x\,(\cos x+\sin x)+C. \qquad (4)$$

Exercise 1. Calculate the following ($a, b \in \mathbb{R}$):

$$\int x\, e^x \cos x\, \mathrm{d}x; \qquad \int e^{ax} \sin b\, x\, \mathrm{d}x. \tag{5}$$

Example 3. Let $n \in \mathbb{N}$. Calculate $\int \cos^n x\, \mathrm{d}x$.

Solution. We have

$$\begin{aligned} \int \cos^n x\, \mathrm{d}x &= \int \cos^{n-1} x\, \mathrm{d}\sin x \\ &= \cos^{n-1} x \sin x - \int \sin x\, \mathrm{d}\cos^{n-1} x \\ &= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x\, \mathrm{d}x \\ &= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\, \mathrm{d}x - (n-1) \int \cos^n x\, \mathrm{d}x. \end{aligned} \tag{6}$$

Now if we denote $I_n := \int \cos^n x\, \mathrm{d}x$ we reach the following formula:

$$I_n = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} I_{n-2}. \tag{7}$$

We see that each time we apply this formula, the power of $\cos x$ is reduced by 2, consequently we can expect to reduce the calculation of $\int \cos^n x\, \mathrm{d}x$ to either $\int \cos x\, \mathrm{d}x$ or $\int \mathrm{d}x$, which are both easy.

For example, to calculate $I_7 = \int \cos^7 x\, \mathrm{d}x$ we have

$$\begin{aligned} I_7 &= \frac{\cos^{7-1} x \sin x}{7} + \frac{7-1}{7} I_{7-2} \\ &= \frac{\cos^6 x \sin x}{7} + \frac{6}{7} I_5 \\ &= \frac{\cos^6 x \sin x}{7} + \frac{6}{7} \left[\frac{\cos^4 x \sin x}{5} + \frac{4}{5} I_3 \right] \\ &= \frac{\cos^6 x \sin x}{7} + \frac{6}{35} \cos^4 x \sin x + \frac{6}{7} \cdot \frac{4}{5} \left[\frac{\cos^2 x \sin x}{3} + \frac{2}{3} I_1 \right] \\ &= \frac{\cos^6 x \sin x}{7} + \frac{6}{35} \cos^4 x \sin x + \frac{8}{35} \cos^2 x \sin x + \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \sin x + C. \end{aligned} \tag{8}$$

Example 4. Calculate $\int \frac{1}{\cos^n x} \mathrm{d}x$.

Solution. We have

$$\begin{aligned} J_n := \int \frac{1}{\cos^n x} \mathrm{d}x &= \int \frac{1}{\cos^{n-2} x} \mathrm{d}\tan x \\ &= \frac{\sin x}{\cos^{n-1} x} - \int \frac{\sin x}{\cos x} \mathrm{d}\left(\frac{1}{\cos^{n-2} x} \right) \\ &= \frac{\sin x}{\cos^{n-1} x} - \int \frac{\sin x}{\cos x} (n-2) \frac{\sin x}{\cos^{n-1} x} \mathrm{d}x \\ &= \frac{\sin x}{\cos^{n-1} x} - (n-2) \int \frac{\sin^2 x}{\cos^n x} \mathrm{d}x \\ &= \frac{\sin x}{\cos^{n-1} x} - (n-2) J_n + (n-2) J_{n-2}. \end{aligned} \tag{9}$$

This gives

$$J_n = \frac{1}{n-1} \frac{\sin x}{\cos^{n-1} x} + \frac{n-2}{n-1} J_{n-2}. \tag{10}$$

Exercise 2. Calculate J_7.

Exercise 3. Calculate $\int \cos^m x \sin^n x \, \mathrm{d}x$ where $m, n \in \mathbb{N}$.

Example 5. Calculate $\int \frac{1}{(x^2+1)^n} \, \mathrm{d}x$.

Solution. We have

$$\begin{aligned} K_n := \int \frac{1}{(x^2+1)^n} \, \mathrm{d}x &= \frac{x}{(x^2+1)^n} - \int x \, \mathrm{d}\left(\frac{1}{(x^2+1)^n} \right) \\ &= \frac{x}{(x^2+1)^n} + 2\,n \int \frac{x^2}{(x^2+1)^{n+1}} \, \mathrm{d}x \\ &= \frac{x}{(x^2+1)^n} + 2\,n\, K_n - 2\,n\, K_{n+1}. \end{aligned} \tag{11}$$

This gives

$$K_{n+1} = \frac{x}{2\,n\,(x^2+1)^n} + \frac{2\,n-1}{2\,n} K_n \tag{12}$$

or equivalently

$$K_n = \frac{x}{2\,(n-1)\,(x^2+1)^{n-1}} + \frac{2\,n-3}{2\,n-2} K_{n-1}. \tag{13}$$

Exercise 4. Calculate K_2, K_3.