Math 118 Winter 2015 Lecture 4 (Jan. 9, 2015)

- Recall

$$
\begin{aligned}
\int x^{\alpha}\,\mathrm{d}x &= \frac{1}{1+\alpha}\,x^{1+\alpha}+C \qquad \alpha\in\mathbb{R}, \quad \alpha\neq 1; && (1)\\
\int \frac{\mathrm{d}x}{x} &= \ln|x|+C; && (2)\\
\int e^{x}\,\mathrm{d}x &= e^{x}+C; && (3)\\
\int \cos x\,\mathrm{d}x &= \sin x+C; && (4)\\
\int \sin x\,\mathrm{d}x &= -\cos x+C; && (5)\\
\int \frac{\mathrm{d}x}{(\cos x)^{2}} &= \tan x+C; && (6)\\
\int \frac{\mathrm{d}x}{(\sin x)^{2}} &= -\cot x+C; && (7)\\
\int \frac{\mathrm{d}x}{\sqrt{1-x^{2}}} &= \arcsin x+C; && (8)\\
\int \frac{\mathrm{d}x}{1+x^{2}} &= \arctan x+C. && (9)
\end{aligned}
$$

- Differentials

  For now we define "differentials" as follows:

  - For the variables $x, t, \ldots$ the differentials are $\mathrm{d}x, \mathrm{d}t, \ldots$
  - For a function $f(x)$, its differential is defined as:

$$\mathrm{d}f := f'(x)\,\mathrm{d}x \tag{10}$$

Example 1. We have

$$\mathrm{d}(\sin x) = \cos x\,\mathrm{d}x, \qquad \mathrm{d}(\tan x) = \frac{\mathrm{d}x}{\cos^{2}x}. \tag{11}$$

Exercise 1. Calculate $\mathrm{d}(\cos x), \mathrm{d}(x^{7}), \mathrm{d}(x-7)$.

Remark 2. It is curious that in fact it does not matter whether x, t, ... is a variable or function. For example, for $f(u)$ we have $\mathrm{d}f = f'(u)\,\mathrm{d}u$. If later we know that u is in fact a function of x, $u(x)$, then we have

$$\mathrm{d}f(u(x)) = f'(u(x))\,u'(x)\,\mathrm{d}x, \qquad f'(u)\,\mathrm{d}u = f'(u(x))\,\mathrm{d}u(x) = f'(u(x))\,u'(x)\,\mathrm{d}x \tag{12}$$

so $\mathrm{d}f = f'(u)\,\mathrm{d}u$ still holds!

With the help of this new symbol "d", we can "streamline" the calculation of integrals.

Example 3. To calculate $\int e^{2x}\,\mathrm{d}x$, we simply write

$$\begin{aligned}
\int e^{2x}\,\mathrm{d}x &= \frac{1}{2}\int e^{2x}\,\mathrm{d}(2\,x) \\
&\overset{u(x)=2x}{=\!=\!=} \frac{1}{2}\int e^{u}\,\mathrm{d}u \\
&= \frac{1}{2}\,e^{u}+C \\
&= \frac{1}{2}\,e^{2x}+C. \qquad (13)
\end{aligned}$$

Note that after one becomes good at manipulating differentials, the greyed steps can be skipped.

Example 4. We have

$$\int \cos^3 x\,\mathrm{d}x = \int \cos^2 x\,\mathrm{d}(\sin x) = \int [1-\sin^2 x]\,\mathrm{d}(\sin x) = \sin x - \frac{\sin^3 x}{3} + C. \qquad (14)$$

- Integrate $\int \frac{\mathrm{d}x}{a\,x^2+b\,x+c}$.
  - Special cases: We consider $a=\pm 1, b=0, c=\pm 1$. There are four combinations:

    $$\int \frac{\mathrm{d}x}{1+x^2}, \quad \int \frac{\mathrm{d}x}{-1-x^2}, \quad \int \frac{\mathrm{d}x}{1-x^2}, \quad \int \frac{\mathrm{d}x}{x^2-1}. \qquad (15)$$

    We have known how to integrate them.
  - General cases.

Example 5. Integrate the following.

a) $\int \frac{\mathrm{d}x}{x^2-x}$;

b) $\int \frac{\mathrm{d}x}{x^2-2\,x-2}$;

c) $\int \frac{\mathrm{d}x}{x^2-4\,x+5}$;

d) $\int \frac{\mathrm{d}x}{x^2-2\,x+1}$.

Solutions.

a) We try to write

$$\frac{1}{x^2-x} = \frac{1}{x\,(x-1)} = \frac{A}{x} + \frac{B}{x-1}. \qquad (16)$$

To figure out A we multiply both sides by x and then set $x = 0$. This gives $A = -1$. Similarly we obtain $B = 1$. Therefore

$$\begin{aligned} \int \frac{\mathrm{d}x}{x^2 - x} &= \int \left[\frac{-1}{x} + \frac{1}{x-1} \right] \mathrm{d}x \\ &= \int \frac{\mathrm{d}x}{x-1} - \int \frac{\mathrm{d}x}{x} \\ &= \int \frac{\mathrm{d}(x-1)}{x-1} - \int \frac{\mathrm{d}x}{x} \\ &= \ln\left| \frac{x-1}{x} \right| + C. \end{aligned} \tag{17}$$

b) We factorize

$$x^2 - 2\,x - 2 = \left(x - \left[1 + \sqrt{3}\right]\right) \left(x - \left[1 - \sqrt{3}\right]\right). \tag{18}$$

Therefore we try to find A, B such that

$$\frac{1}{\left(x - \left[1 + \sqrt{3}\right]\right) \left(x - \left[1 - \sqrt{3}\right]\right)} = \frac{A}{x - \left[1 + \sqrt{3}\right]} + \frac{B}{x - \left[1 - \sqrt{3}\right]}. \tag{19}$$

Multiply both sides by $x - \left[1 - \sqrt{3}\right]$ and setting $x = 1 - \sqrt{3}$ we have $B = -\frac{1}{2\sqrt{3}}$ and similarly we have $A = \frac{1}{2\sqrt{3}}$. Thus

$$\begin{aligned} \int \frac{\mathrm{d}x}{x^2 - 2\,x - 2} &= \frac{1}{2\sqrt{3}} \left[\int \frac{\mathrm{d}x}{x - \left[1 + \sqrt{3}\right]} - \int \frac{\mathrm{d}x}{x - \left[1 - \sqrt{3}\right]} \right] \\ &= \frac{1}{2\sqrt{3}} \ln\left| \frac{x - \left[1 + \sqrt{3}\right]}{x - \left[1 - \sqrt{3}\right]} \right| + C. \end{aligned} \tag{20}$$

c) We notice that $x^2 - 4\,x + 5$ cannot be factorized. But whenever this happens, it could be written as a sum of squares:

$$x^2 - 4\,x + 5 = (x-2)^2 + 1. \tag{21}$$

Therefore

$$\begin{aligned} \int \frac{\mathrm{d}x}{x^2 - 4\,x + 5} &= \int \frac{\mathrm{d}x}{(x-2)^2 + 1} \\ &= \int \frac{\mathrm{d}(x-2)}{(x-2)^2 + 1} \\ &= \arctan(x-2) + C. \end{aligned} \tag{22}$$

d) We have

$$\int \frac{\mathrm{d}x}{x^2 - 2\,x + 1} = \int \frac{\mathrm{d}(x-1)}{(x-1)^2} = -\frac{1}{x-1} + C. \tag{23}$$

Exercise 2. Use a random integer generator (or a friend) to obtain 10 similar problems. Calculate the indefinite integrals and compare with the results given by www.wolframalpha.com.

Exercise 3. Calculate $\int \frac{\mathrm{d}x}{a\,x^2 + b\,x + c}$ where $a, b, c \in \mathbb{R}$ and at least one is nonzero.

Exercise 4. Consider $\int \frac{\mathrm{d}x}{a\,x^2 + b\,x + c}$ where $a\,x^2 + b\,x + c$ has two distinct real roots. Explain how to calculate this integral by reducing it to $\int \frac{\mathrm{d}x}{x^2 - 1}$ through appropriate substitution.

- Integrate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{a\,x^2+b\,x+c}}$.
  - Special cases: Again there are four of them:

$$\int \frac{\mathrm{d}x}{\sqrt{1-x^2}}, \quad \int \frac{\mathrm{d}x}{\sqrt{1+x^2}}, \quad \int \frac{\mathrm{d}x}{\sqrt{x^2-1}}, \quad \int \frac{\mathrm{d}x}{\sqrt{-x^2-1}}. \tag{24}$$

We see that the first one is in the table, and the last one is meaningless.

Example 6. Calculate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{1+x^2}}$.

Solution. We have

$$\begin{aligned}
\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} &\overset{x=\tan t}{=\!=} \int \frac{\frac{1}{\cos^2 t}\,\mathrm{d}t}{\sqrt{\frac{1}{\cos^2 t}}} && (25)\\
&= \int \frac{\mathrm{d}t}{\cos t} && (26)\\
&= \int \frac{\cos t\,\mathrm{d}t}{\cos^2 t}\\
&= \int \frac{\mathrm{d}(\sin t)}{1-\sin^2 t}\\
&\overset{u=\sin t}{=\!=} \frac{1}{2}\left[\int \frac{\mathrm{d}u}{1-u} + \int \frac{\mathrm{d}u}{1+u}\right]\\
&= \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + C. && (27)
\end{aligned}$$

To obtain the final answer, we notice

$$\frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| = \frac{1}{2}\ln\left|\frac{(1+\sin t)^2}{1-\sin^2 t}\right| = \ln\left|\frac{1+\sin t}{\cos t}\right| = \ln\left(x+\sqrt{1+x^2}\right). \tag{28}$$

Therefore

$$\int \frac{\mathrm{d}x}{\sqrt{1+x^2}} = \ln\left(x+\sqrt{1+x^2}\right) + C. \tag{29}$$

Exercise 5. Explain how does (26) follow from (25). (Hint:[1])

Exercise 6. Obtain the final result from (27) by explicitly finding the relation $u = u(x)$.

Exercise 7. Calculate $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-1}}$. (Hint:[2])

Exercise 8. Explain how to calculate

$$\int \frac{\mathrm{d}x}{\sqrt{a\,x^2+b\,x+c}} \tag{30}$$

for general $a, b, c \in \mathbb{R}$. How many different cases are there?

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1. When we set $x = \tan t$, what interval is t in?
2. $x = \sec t$.