MATH 117 FALL 2014 LECTURE 48 (DEC. 3, 2014)

- Higher derivatives.
  - Leibniz formula:
$$(f\,g)^{(n)}=\sum_{k=0}^{n}\binom{n}{k}f^{(k)}\,g^{(n-k)}. \tag{1}$$

    Example 1. Calculate $(x\sin x)^{(100)}$.
    Solution. We notice that $x^{(k)}=0$ for all $k\geqslant 2$. Thus
$$(x\sin x)^{(100)}=\binom{100}{0}x\,(\sin x)^{(100)}+\binom{100}{1}x^{(1)}\,(\sin x)^{(99)}=x\sin x-100\cos x. \tag{2}$$

- L'Hospital.
  - Four assumptions and one conclusion:
    - A1: $\lim_{x\to a}f(x)=\lim_{x\to a}g(x)=0$ (or $\pm\infty$);
    - A2: f,g differentiable on $(a-\delta,a)\cup(a,a+\delta)$ for some $\delta>0$;
    - A3: $\lim_{x\to a}\frac{f'(x)}{g'(x)}=L$;
    - A4: $g'\neq 0$ on $(a-\delta,a)\cup(a,a+\delta)$ for some $\delta>0$.
    - C1: If A1 – A4 are satisfied, then
$$\lim_{x\to a}\frac{f(x)}{g(x)}=L. \tag{3}$$
  - Note that a,L could be either real numbers or $\pm\infty$; Also $x\to a$ could be replaced by $x\to a+$ or $x\to a-$.

Example 2. Let $f(x):=\begin{cases}e^{-1/x} & x>0\\ 0 & x\leqslant 0\end{cases}$. Calculate $f'(0)$, $f''(0)$.

Solution.

- $f'(0)$.
  Clearly $\lim_{x\to 0-}\frac{f(x)-f(0)}{x-0}=0$. On the other hand, we have
$$\begin{aligned}\lim_{x\to 0+}\frac{f(x)-f(0)}{x-0} &= \lim_{x\to 0+}\frac{e^{-1/x}}{x}\\ &= \lim_{t\to+\infty}\frac{e^{-t}}{1/t}\\ &= \lim_{t\to+\infty}\frac{t}{e^t}\\ &= \lim_{t\to+\infty}\frac{1}{e^t}=0.\end{aligned}$$

  Therefore $f'(0)=0$.

  Exercise 1. Prove by definition $\lim_{x\to 0+}\frac{e^{-1/x}}{x}=\lim_{t\to+\infty}\frac{e^{-t}}{1/t}$.

  Exercise 2. Explain why direct application of L'Hospital to $\lim_{x\to 0+}\frac{e^{-1/x}}{x}$ does not work.

- $f''(0)$.

  We calculate

$$f'(x) = \begin{cases} \frac{1}{x^2} e^{-1/x} & x > 0 \\ 0 & x \leqslant 0 \end{cases}. \tag{4}$$

Exercise 3. Apply L'Hospital to prove $\lim_{x \to 0+} \frac{f'(x) - f'(0)}{x - 0} = 0$.

- Taylor expansion.

  - If $f(x_0)$, $f'(x_0)$, ..., $f^{(n)}(x_0)$ exist, then we can define the Taylor polynomial of f at x_0 to degree n:

$$T_n(x) = f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \tag{5}$$

Then the remainder is small compared to $(x - x_0)^n$:

$$\lim_{x \to x_0} \frac{f(x) - T_n(x)}{(x - x_0)^n} = 0. \tag{6}$$

  - If we make the stronger assumption: $f(x)$, ..., $f^{(n+1)}(x)$ exist on $(a, b) \ni x_0$, then for every $x \in (a, b)$ there exists $c \in (x_0, x)$:

$$f(x) - T_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \tag{7}$$

Remark 3. Note that this gives

$$\frac{f(x) - T_n(x)}{(x - x_0)^n} = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0) \tag{8}$$

and now we know much more precisely how small the remainder $f(x) - T_n(x)$ is compared to $(x - x_0)^n$, as long as we have some idea of $\sup_{c \in (x_0, x)} \left| f^{(n+1)}(c) \right|$.

Example 4. Estimate $\left| \cos x - \left(1 - \frac{x^2}{2}\right) \right|$ for $x = 10^{-1}$.

Solution. We notice $1 - \frac{x^2}{2}$ is the Taylor polynomial of $\cos x$ at $x_0 = 0$ to degree 3. Therefore

$$\left| \cos x - \left(1 - \frac{x^2}{2}\right) \right| = \left| \frac{f^{(4)}(c)}{4!} x^4 \right| \leqslant \frac{|x|^4}{24}. \tag{9}$$

Setting $x = 10^{-1}$ we see

$$\left| \cos\left(\frac{1}{10}\right) - 0.995 \right| \leqslant \frac{1}{240000} \approx 4.17 \times 10^{-6}. \tag{10}$$

Remark 5. We can check that $\left| \cos\left(\frac{1}{10}\right) - 0.995 \right| \approx 4.17 \times 10^{-6}$. So our estimate is very accurate.

- Power series.

Example 6. Find all $x \in \mathbb{R}$ such that $\sum_{n=1}^{\infty} n x^n$ is convergent.

Solution. We calculate the radius of convergence:

$$\rho := \left(\limsup_{n\to\infty} n^{1/n} \right) = 1. \tag{11}$$

Therefore the power series is convergent for $|x| < 1$ and divergent for $|x| > 1$.

Next we check the convergence/divergence at $x = \pm 1$. For such x we have $\lim_{n\to\infty} n\, x^n = 0$ does not hold and therefore the power series diverges.

Summarizing, we see that the power series converges for $|x| < 1$ and diverges for $|x| \geqslant 1$.