# Math 117 Fall 2014 Lecture 47 (Dec. 1, 2014)

- Proving integrability.
  - Definition: $\inf_P U(f,P) = \sup_P L(f,P)$.
  - If there is $\{P_n\}$ such that $\lim_{n\to\infty} U(f,P_n) = \lim_{n\to\infty} L(f,P_n)$.
  - If there is $\{P_n\}$ such that $\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0$.

**Example 1.** Prove: If there is $\{P_n\}$ such that $\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0$, then $f$ is integrable.

**Proof.** By definition we have

$$U(f,P_n) \geqslant U(f), \quad L(f,P_n) \leqslant L(f) \Longrightarrow U(f,P_n) - L(f,P_n) \geqslant U(f) - L(f). \tag{1}$$

On the other hand we know $U(f) \geqslant L(f)$. Applying Comparison Theorem to

$$U(f,P_n) - L(f,P_n) \geqslant U(f) - L(f) \geqslant 0 \tag{2}$$

we have $0 \geqslant U(f) - L(f) \geqslant 0$ which gives $U(f) = L(f)$ and integrability. □

**Example 2.** Let $f: [a,b] \mapsto \mathbb{R}$ be increasing. Then $f$ is integrable on $[a,b]$.

**Proof.** Let $x_k := a + \frac{b-a}{n} \cdot k$ and take $P_n = \{x_0, \ldots, x_n\}$. Then we have

$$\begin{aligned} U(f,P_n) - L(f,P_n) &= \sum_{k=1}^{n} \left[ \sup_{[x_{k-1},x_k]} f - \inf_{[x_{k-1},x_k]} f \right] (x_k - x_{k-1}) \\ &= \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] \frac{b-a}{n} \\ &= \frac{b-a}{n} \sum_{k=1}^{n} [f(x_k) - f(x_{k-1})] \\ &= \frac{b-a}{n} [f(x_n) - f(x_0)] = \frac{(b-a)(f(b) - f(a))}{n}. \end{aligned} \tag{3}$$

Thus $\lim_{n\to\infty} [U(f,P_n) - L(f,P_n)] = 0$ and integrability follows. □

**Exercise 1.** Let $f: [a,b] \mapsto \mathbb{R}$ be decreasing. Then $f$ is integrable on $[a,b]$.

**Remark 3.** Let $f: [a,b] \mapsto \mathbb{R}$ be increasing. It turns out that it can at most be discontinuous at countably many points. An example is as follows. Let $\mathbb{Q} \cap [a,b] = \{r_1, r_2, \ldots\}$. Define

$$f(x) := \sum_{k=1}^{\infty} 2^{-k} \chi_{[r_k,b]}(x). \tag{4}$$

where $\chi_{[r,b]}(x) := \begin{cases} 1 & x \geqslant r \\ 0 & x < r \end{cases}$.

**Problem 1.** Prove that $f(x)$ is increasing and discontinuous at every rational point but continuous at every irrational point.

- Fundamental Theorems of Calculus.
  - FTC1.
    - Calculation of $\int_a^b f(x)\,\mathrm{d}x$.

- Key: Find $F(x)$ continuous on $[a, b]$ such that $F'(x) = f(x)$ on $(a, b)$.

**Example 4.** Let $F(x) = \begin{cases} x^2 \sin\frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Then $F(x)$ is differentiable everywhere, but $f(x) = F'(x)$ is not integrable on $[0, 1]$, as $f(x)$ is not bounded on this interval.

- FTC2.
  - Note that only when $f(x)$ is continuous at $c$ is $G(x) = \int_a^x f(t)\, \mathrm{d}t$ is differentiable at $c$ and such that $G'(c) = f(c)$.

**Example 5.** Let $f(x)$ be continuous on $[a, b]$. Then for any $c, d \in [a, b]$ we have

$$\int_c^d f(x)\, \mathrm{d}x = F(d) - F(c) \tag{5}$$

where $F$ is any anti-derivative of $f$, that is $F'(x) = f(x)$ on $(a, b)$. Furthermore $G(x) := \int_a^x f(t)\, \mathrm{d}t$ is also an anti-derivative of $f$.