Math 117 Fall 2014 Lecture 44 (Nov. 26, 2014)

Read: Bowman §4.G; **314** Differentiation §4.2.

- Taylor Expansion.

  Let $n \in \mathbb{N} \cup \{0\}$, $x_0 \in \mathbb{R}$, $f: \mathbb{R} \mapsto \mathbb{R}$. Define

  - Taylor polynomial of f at x_0 of degree n:

$$T_n(x) := f(x_0) + f'(x_0)\,(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}\,(x - x_0)^n. \tag{1}$$

  - The "remainder" term:

$$R_n(x) := f(x) - T_n(x). \tag{2}$$

Theorem 1. (Taylor Expansion with Peano Form of Remainder) *If $f^{(n)}(x_0)$ exists, then*

$$\lim_{x \to x_0} \frac{R_n(x)}{(x - x_0)^n} = 0. \tag{3}$$

Remark 2. Note that the existence of $f^{(n)}(x_0)$ is necessary for $T_n(x)$ be defined so the hypothesis is already minimal.

Remark 3. In the case $n = 0$ Theorem 1 becomes $\lim_{x \to x_0} [f(x) - f(x_0)] = 0$ which is simply continuity of f at x_0.

Exercise 1. Let $n \in \mathbb{N}$. Prove or disprove: If there is a polynomial P_n of degree n such that $\lim_{x \to 0} \frac{f(x) - P_n(x)}{x^n} = 0$, then $f^{(n)}(0)$ exists. (Hint:[1])

Theorem 4. (Taylor Expansion with Lagrange Form of Remainder) *If $f, f', \ldots, f^{(n)}$ are continuous on $[x_0, x]$*[2] *and $f^{(n)}$ is differentiable on (x_0, x), then there is $c \in (x_0, x)$ such that*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x - x_0)^{n+1}. \tag{4}$$

Remark 5. Note that the assumption is also minimal already.

We give two proofs for the case $n = 1$ and leave the case of general n as exercises. We need to show there is $c \in (x_0, x)$ such that

$$f(x) = T_1(x) + \frac{f''(c)}{2}\,(x - x_0)^2. \tag{5}$$

Proof. (Normal) Fix x_0, x. Then there is $r \in \mathbb{R}$ such that $R_n(x) = r\,(x - x_0)^2$.

Define

$$\phi(t) := f(t) - [T_1(t) + r\,(t - x_0)^2]. \tag{6}$$

We check $\phi(x) = \phi(x_0) = 0$. Application of Rolle's Theorem yields the existence of $c_1 \in (x_0, x)$ such that $\phi'(c_1) = 0$. As $\phi'(x_0) = 0$ we apply Rolle's Theorem again for ϕ' on (x_0, c) and obtain $\phi''(c) = 0$ for some $c \in (x_0, c) \subset (x_0, x)$ which gives $2\,r = f''(c)$ and the proof ends. ☐

1. Consider $x^{n+1}\,D(x)$ where $D(x)$ is the Dirichlet function.
2. If $x < x_0$ this is understood as $[x, x_0]$.

Proof. (CLEVER) Fix x_0, x. Define

$$F(t) := f(x) - [f(t) + f'(t)(x - t)]; \qquad G(t) := (x - t)^2. \tag{7}$$

Application of Cauchy's MVT on $[x_0, x]$ (t is the variable here) gives

$$\frac{R(x)}{(x - x_0)^2} = \frac{F(x_0) - F(x)}{G(x_0) - G(x)} = \frac{F'(c)}{G'(c)} = \frac{f''(c)}{2} \tag{8}$$

for some $c \in (x_0, x)$ and the proof ends. □

- Example.

Example 6. Calculate the generic of Taylor expansion of e^x at $x_0 = 0$ with Lagrange form of remainder.

As $(e^x)^{(n)} = e^x$ for every n, we have

$$e^x = T_n(x) + R_n(x) = 1 + x + \cdots + \frac{x^n}{n!} + \frac{e^c}{(n+1)!}\, x^{n+1}. \tag{9}$$

Note that as e^x is n-th differentiable at every x for every n, (9) holds for every $x \in \mathbb{R}$ and every $n \in \mathbb{N} \cup \{0\}$.