Math 117 Fall 2014 Lecture 43 (Nov. 24, 2014)

Read: Bowman §4.G; **314** Differentiation §4.2.

- Continuity and Differentiability as Approximations.

Example 1. Let $f(x)$ be continuous at x_0. Then there is exactly one number $s_0 \in \mathbb{R}$ such that $\lim_{x \to x_0} [f(x) - s_0] = 0$.

Exercise 1. Prove that $s_0 = f(x_0)$.

Exercise 2. Prove that $T_0(x) = f(x_0)$ is the best approximation of $f(x)$ at x_0 by polynomials of degree zero, in the following sense:

Let $P_0(x)$ be any other zeroth degree polynomial, there holds

$$\lim_{x \to x_0} \frac{|f(x) - T_0(x)|}{|f(x) - P_0(x)|} = 0. \tag{1}$$

Exercise 3. Prove that

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0 \Longleftrightarrow \lim_{x \to x_0} \frac{|f(x)|}{|g(x)|} = 0. \tag{2}$$

Example 2. Let $f(x)$ be differentiable at x_0. Then there are exactly two numbers $s_0, s_1 \in \mathbb{R}$ such that $\lim_{x \to x_0} \frac{f(x) - [s_0 + s_1 (x - x_0)]}{x - x_0} = 0$.

Exercise 4. Prove that $s_0 = f(x_0)$, $s_1 = f'(x_0)$.

Exercise 5. Prove that $T_1(x) = f(x_0) + f'(x_0)\,(x - x_0)$ is the best approximation of $f(x)$ at x_0 by polynomials of degree at most one, in the following sense:

Let $P_1(x)$ be any other polynomial of degree at most one, there holds

$$\lim_{x \to x_0} \frac{|f(x) - T_1(x)|}{|f(x) - P_1(x)|} = 0. \tag{3}$$

(Note:[1])

- Generalization.
  - Approximation by polynomials of degree up to two.

Example 3. Let $f(x)$ be twice differentiable at x_0. Then there are exactly three numbers $s_0, s_1, s_2 \in \mathbb{R}$ such that

$$\lim_{x \to x_0} \frac{f(x) - [s_0 + s_1 (x - x_0) + s_2 (x - x_0)^2]}{(x - x_0)^2} = 0. \tag{5}$$

We claim that $s_0 = f(x_0), s_1 = f'(x_0), s_2 = \frac{f''(x_0)}{2}$. To see this, first we show that these are the only possibile values.

As $\lim_{x \to x_0} (x - x_0)^2 = 0$, it is necessarily that $\lim_{x \to x_0} \{f(x) - [s_0 + s_1 (x - x_0) + s_2 (x - x_0)^2]\} = 0$ which implies $s_0 = f(x_0)$.

Next we have

$$\frac{f(x) - [f(x_0) + s_1 (x - x_0) + s_2 (x - x_0)^2]}{(x - x_0)^2} = \frac{\frac{f(x) - f(x_0)}{x - x_0} - [s_1 + s_2 (x - x_0)]}{(x - x_0)}. \tag{6}$$

1. The following proof is not correct for the case $P_1(x_0) = f(x_0)$ (Sorry!). By L'Hospital we have

$$\lim_{x \to x_0} \frac{f(x) - T_1(x)}{f(x) - P_1(x)} = \lim_{x \to x_0} \frac{[f(x) - T_1(x)]'}{[f(x) - P_1(x)]'} = \lim_{x \to x_0} \frac{f'(x) - f'(x_0)}{f'(x) - P_1'(x_0)} = \frac{0}{f'(x_0) - P_1'(x_0)} = 0. \tag{4}$$

Taking limit $x \to x_0$ we see that the numerator tends to $f'(x_0) - s_1$ while the denominator tends to 0 and that (5) holds implies $s_1 = f'(x_0)$.

Now as $f''(x_0)$ exists, there is $\delta > 0$ such that $f'(x)$ exists on $(x_0 - \delta, x_0 + \delta)$. Applying L'Hospital on this interval we obtain

$$\begin{aligned}
&\lim_{x \to x_0} \frac{f(x) - [f(x_0) + f'(x_0)(x - x_0) + s_2 (x - x_0)^2]}{(x - x_0)^2} \\
= &\lim_{x \to x_0} \frac{\{f(x) - [f(x_0) + f'(x_0)(x - x_0) + s_2 (x - x_0)^2]\}'}{\{(x - x_0)^2\}'} \\
= &\lim_{x \to x_0} \frac{\{f'(x) - [f'(x_0) + 2 s_2 (x - x_0)]\}}{2 (x - x_0)} \\
= &\lim_{x \to x_0} \frac{\frac{f'(x) - f'(x_0)}{x - x_0} - 2 s_2}{2} \\
= &\frac{f''(x_0) - 2 s_2}{2}. \qquad (7)
\end{aligned}$$

We see that necessarily $s_2 = \frac{f''(x_0)}{2}$.

Exercise 6. Prove that (5) holds for $s_0 = f(x_0), s_1 = f'(x_0), s_2 = \frac{f''(x_0)}{2}$.

Exercise 7. Prove that $f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2$ is the best approximation of $f(x)$ at x_0 by polynomials of degree up to two.

- Taylor's Theorem.

THEOREM 4. *Let $f(x)$ be n-th differentiable at x_0. Then there exist exactly $n+1$ real numbers $s_0, \ldots, s_n$ such that*

$$\lim_{x \to x_0} \frac{f(x) - [s_0 + s_1 (x - x_0) + \cdots + s_n (x - x_0)^n]}{(x - x_0)^n} = 0. \qquad (8)$$

Furthermore $s_0 = f(x_0), s_1 = f'(x_0), \ldots, s_n = \frac{f^{(n)}(x_0)}{n!}$.

Exercise 8. Prove Theorem 4.

DEFINITION 5. *Let $f(x)$ be n-th differentiable at x_0. Define the Taylor polynomial of degree n of $f(x)$ at x_0 as*

$$T_n(x) := f(x_0) + f'(x_0)(x - x_0) + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \qquad (9)$$

Define the "remainder" as $R_n(x) := f(x) - T_n(x)$.

Remark 6. Note that $T_n(x)$ depends on 1. n, 2. $f(x)$, 3. x_0.

Remark 7. $R_n(x)$ describes how well f is approximated by $T_n(x)$.

THEOREM 8. (TAYLOR EXPANSION WITH LANGRANGE FORM OF REMAINDER) *Let $f(x)$ be $(n+1)$-th differentiable on (a,b) and $x_0 \in (a,b)$. Then there is $c \in (a,b)$ such that*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \qquad (10)$$

Proof. Next lecture. □

Exercise 9. Detect the mistake in the following "proof" of Theorem 8 in the case $n=2$: Apply MVT to $f''(x)$ between x_0 and t, where t is arbitrary and between x_0, x, we have for some c,

$$f''(t) - f''(x_0) = f'''(c)\,(t - x_0). \tag{11}$$

Integrating from x_0 to u with respect to t we have

$$f'(u) - f'(x_0) - f''(x_0)\,(u - x_0) = \frac{f'''(c)}{2}\,(u - x_0)^2. \tag{12}$$

Integrating again from x_0 to x with respect to u we have

$$f(x) - f(x_0) - f'(x_0)\,(x - x_0) - \frac{f''(x_0)}{2}\,(x - x_0)^2 = \frac{f'''(c)}{6}\,(x - x_0)^3 \tag{13}$$

and our proof ends.