Math 117 Fall 2014 Midterm Exam 3 Solutions

Nov. 21, 2014 10am - 10:50am. Total 20+2 Pts

NAME: ID#:

- There are five questions.
- Please write clearly and show enough work.

1	
2	
3	
4	
5	
Total	

Question 1. (5 pts) *Prove by* $\varepsilon-\delta$*:* $f(x) := \begin{cases} 2 & x>0 \\ 1 & x \leqslant 0 \end{cases}$ *is continuous at every* $a \neq 0$ *but discontinuous at* 0*.*

Proof. Let $a \neq 0$ be arbitrary. Let $\varepsilon > 0$ be arbitrary. Take $\delta = |a|$. Then for every $|x-a| < \delta$, we have either both $x, a > 0$ or both $x, a < 0$. Consequently

$$|f(x) - f(a)| = 0 < \varepsilon \tag{1}$$

and continuity follows.

At 0, let $\delta > 0$ be arbitrary. Take $x \in (0, \delta)$. Then we have

$$|f(x) - f(0)| = 1 \tag{2}$$

and discontinuity follows. □

Question 2. (5 pts) *Let* $f(x) := \begin{cases} x + x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. *Prove that f is differentiable everywhere on $\mathbb{R}$ and calculate $f'(x)$.*

Proof. Since $x, x^2, \sin x$ are differentiable everywhere and $1/x$ is differentiable everywhere except at 0, we have $x + x^2 \sin\frac{1}{x}$ differentiable at every $x \neq 0$. We further calculate for $x \neq 0$,

$$f'(x) = 1 + \left(x^2 \sin\frac{1}{x}\right)' = 1 + 2x \sin\frac{1}{x} - \cos\frac{1}{x}. \tag{3}$$

At 0, we have

$$\frac{f(x) - f(0)}{x - 0} = 1 + x \sin\frac{1}{x}. \tag{4}$$

Following Squeeze Theorem we have $\lim_{x \to 0} x \sin\frac{1}{x} = 0$ and consequently

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = 1 \tag{5}$$

and there follows $f'(0) = 1$.

Summarize:

$$f'(x) = \begin{cases} 1 + 2x \sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}. \tag{6}$$

Thus ends the solution. □

Question 3. (5 pts) *Prove or disprove:* $\sum_{n=1}^{\infty} \tan\frac{1}{n^2}$ *converges. (You can use the convergence/divergence of* $\sum_{n=1}^{\infty} \frac{1}{n^a}$ *without justification)*

Solution. It converges. Since for every $x \in (0, 1)$, $\sin x < x$ and $\cos x > \cos 1$, we have

$$\forall n \in \mathbb{N}, \qquad \left|\tan\frac{1}{n^2}\right| = \frac{\sin(1/n^2)}{\cos(1/n^2)} < \frac{1/n^2}{\cos 1} = \frac{1}{\cos 1}\frac{1}{n^2}. \tag{7}$$

As $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so does $\frac{1}{\cos 1}\frac{1}{n^2}$ and our conclusion follows from Comparison.

Question 4. (5 pts) *Prove that there are exactly two solutions for the equation* $x^2+1=2\cos x$.

Proof. Let $f(x):=x^2+1-2\cos x$. Clearly f is continuous and differentiable on $\mathbb{R}$.

We calculate $f(-1)=f(1)=2-2\ \cos 1>0$, $f(0)=1-2=-1<0$. Application of IVT on $[-1,0]$ and $[0,1]$ yields two solutions c_1, c_2 such that

$$c_1\in(-1,0),\quad c_2\in(0,1). \tag{8}$$

To see that they are the only solutions, first notice that when $|x|>1$, we have

$$f(x)>2-2\cos x>0 \tag{9}$$

therefore no solution can be outside $(-1,1)$.

Next we prove that f is strictly increasing on $(0,1)$ and strictly decreasing on $(-1,0)$.

To do this we calculate

$$f'(x)=2\,(x+\sin x). \tag{10}$$

As $\sin x<0$ for $x\in(-1,0)$ and $\sin x>0$ for $x\in(0,1)$ we see that $f'(x)<0$ on $(-1,0)$ and >0 on $(0,1)$.

Thus $f(x)=0$ has exactly one solution in $(0,1)$ and one solution in $(-1,0)$. As $f(0)\neq 0$, we finished our proof. □

Question 5. (Extra 2 pts) *Find a function $f: \mathbb{R} \mapsto \mathbb{R}$ such that f is differentiable everywhere, $f'(0) > 0$, but there is no $\delta > 0$ such that f is increasing on $(-\delta, \delta)$. Justify.*

Solution. We consider

$$f(x) = k\,x + x^2 \sin\left(\frac{1}{x}\right). \tag{11}$$

We have

$$f'(x) = \begin{cases} k - \cos\left(\frac{1}{x}\right) + 2\,x \sin\left(\frac{1}{x}\right) & x \neq 0 \\ k & x = 0 \end{cases}. \tag{12}$$

Thus $f'(0) > 0$ as long as $k > 0$.

We prove

> If $k \leqslant 1$ then f is not increasing on any (a, b) containing 0; On the other hand, if $k > 1$ then there is a small interval containing 0 such that f is increasing.

- $k \leqslant 1$. All we need to do is to show that there are $a_n < b_n$, $a_n, b_n \longrightarrow 0$ such that $f'(x) < 0$ for $x \in (a_n, b_n)$.

  We have

  $$f'(x) = k - \sqrt{1 + 4\,x^2} \cos\left(\frac{1}{x} + \theta(x)\right) \tag{13}$$

  for $\theta(x)$ satisfying $\tan(\theta) = 2\,x$. Thus $\theta(x)$ is differentiable and $\theta(x) \longrightarrow 0$ as $x \longrightarrow 0$. As $\frac{1}{x} \longrightarrow \infty$ when $x \longrightarrow \infty$, there are $x_n \longrightarrow 0$ such that

  $$\frac{1}{x_n} + \theta(x_n) = 2\,n\,\pi; \tag{14}$$

  Now as

  $$f'(x_n) = k - \sqrt{1 + 4\,x_n^2} < 0 \tag{15}$$

  there is $\delta_n > 0$ such that

  $$f'(x) < 0 \qquad \forall x \in (x_n - \delta_n, x_n + \delta_n) \tag{16}$$

  thanks to the continuity of $f'(x)$ for $x > 0$.

- $k > 1$. In this case set $\delta := \frac{\sqrt{k-1}}{2}$. Then we have, for all $x \in (-\delta, \delta)$,

  $$f'(x) \geqslant k - \sqrt{1 + 4\,x^2} \geqslant k - (1 + 2\,x^2) \geqslant k - (1 + 2\,\delta^2) = \frac{k-1}{2} > 0. \tag{17}$$

  Therefore f is increasing in $(-\delta, \delta)$.