Math 117 Fall 2014 Homework 9 Solutions

Due Thursday Nov. 27 3pm in Assignment Box

Question 1. (5 pts) *Prove **by definition** that* $f(x) = \begin{cases} 0 & x \neq 0 \\ 1 & x = 0 \end{cases}$ *is integrable on* $[0,1]$.

Proof. Let $P = \{x_0, x_1, \ldots, x_n\}$ be an arbitrary partition of $[0,1]$ with $0 = x_0 < x_1 < \cdots < x_n = 1$. We calculate

$$U(f,P) = \left(\sup_{[x_0,x_1]} f\right)(x_1 - x_0) + \cdots + \left(\sup_{[x_{n-1},x_n]} f\right)(x_n - x_{n-1}) = x_1 - x_0; \tag{1}$$

$$L(f,P) = \left(\inf_{[x_0,x_1]} f\right)(x_1 - x_0) + \cdots + \left(\inf_{[x_{n-1},x_n]} f\right)(x_n - x_{n-1}) = 0. \tag{2}$$

As P is arbitrary, clearly $L(f) = 0$. On the other hand,

$$\inf_{0 = x_0 < x_1 < \cdots < x_n = 1} x_1 - x_0 = 0 \tag{3}$$

so $U(f) = 0$. Therefore f is integrable on $[0,1]$. □

Question 2. (5 pts) *Let* $f: [a,b] \mapsto \mathbb{R}$ *be integrable. Prove that* $|f(x)|$ *is also integrable on* $[a,b]$ *and furthermore* $\left|\int_a^b f(x)\,\mathrm{d}x\right| \leqslant \int_a^b |f(x)|\,\mathrm{d}x$.

Proof. Let $a, b \in \mathbb{R}$. By triangle inequality we have

$$|a| \leqslant |-b| + |a + (-b)| = |b| + |a - b| \tag{4}$$

and similarly $|b| \leqslant |a| + |a - b|$. Therefore $||a| - |b|| \leqslant |a - b|$.

Now let P be an arbitrary partition of $[a,b]$ with $P: a = x_0 < x_1 < \cdots < x_n = b$. For an arbitrary $k \in \{1, 2, \ldots, n\}$ we have

$$\forall x, y \in [x_{k-1}, x_k], \qquad ||f(x)| - |f(y)|| \leqslant |f(x) - f(y)|. \tag{5}$$

Now let x_n, y_n be such that $\lim_{n\to\infty} |f(x_n)| = \sup_{[x_{k-1},x_k]} |f(x)|$ and $\lim_{n\to\infty} |f(y_n)| = \inf_{[x_{k-1},x_k]} |f(x)|$. We conclude

$$\sup_{[x_{k-1},x_k]} |f(x)| - \inf_{[x_{k-1},x_k]} |f(x)| = \lim_{n\to\infty} ||f(x_n)| - |f(y_n)|| \leqslant \limsup_{n\to\infty} |f(x_n) - f(y_n)| \leqslant \sup_{[x_{k-1},x_k]} f(x) - \inf_{[x_{k-1},x_k]} f(x). \tag{6}$$

From this it is now clear that

$$\begin{aligned} U(|f|,P) - L(|f|,P) &= \sum_{k=1}^{\infty} \left(\sup_{[x_{k-1},x_k]} |f(x)| - \inf_{[x_{k-1},x_k]} |f(x)|\right)(x_k - x_{k-1}) \\ &\leqslant \sum_{k=1}^{\infty} \left(\sup_{[x_{k-1},x_k]} f(x) - \inf_{[x_{k-1},x_k]} f(x)\right)(x_k - x_{k-1}) \\ &= U(f,P) - L(f,P). \end{aligned} \tag{7}$$

As f is integrable there is P_n such that $\lim_{n\to\infty} U(f,P) - L(f,P) = 0$. For the same $\{P_n\}$ we have $\lim_{n\to\infty} [U(|f|,P) - L(|f|,P)] = 0$ and consequently $|f|$ is also integrable on $[a,b]$.

Now let P again by arbitrary. By triangle inequality we have

$$|U(f,P)| = \left| \sum_{k=1}^{n} \left(\sup_{[x_{k-1},x_k]} f(x) \right) (x_k - x_{k-1}) \right| \leqslant \sum_{k=1}^{n} \left(\sup_{[x_{k-1},x_k]} |f(x)| \right) (x_k - x_{k-1}) = U(|f|,P), \tag{8}$$

Thus

$$\int_a^b |f(x)|\,\mathrm{d}x = \sup_P U(|f|,P) \geqslant \sup_P |U(f,P)| \geqslant \left| \sup_P U(f,P) \right| = \left| \int_a^b f(x)\,\mathrm{d}x \right| \tag{9}$$

and the proof ends. □

Question 3. (5 pts) *Let the "Naive Integral" of $f: [a,b] \mapsto \mathbb{R}$ be defined as*

$$\mathcal{NI}(f,[a,b]) := \lim_{n\to\infty} \frac{b-a}{n} \sum_{k=1}^{n} f(x_k) \tag{10}$$

where $x_k := a + \frac{k}{n}(b-a)$. Find real numbers $a < c < b$ and a function $f: [a,b] \mapsto \mathbb{R}$ such that $\mathcal{NI}(f,[a,b]) \neq \mathcal{NI}(f,[a,c]) + \mathcal{NI}(f,[c,b])$. Justify your example and explain why we did not define integrals using (10).

Solution. Let $f := \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$. Let $a = 0, b = 1, c = \sqrt{2}$. Then for every $n, k \in \mathbb{N}$, we have

$$0 + \frac{k}{n}(1-0) \in \mathbb{Q}, \quad 0 + \frac{k}{n}\left(\sqrt{2} - 0\right) \notin \mathbb{Q}, \quad 1 + \frac{k}{n}\left(\sqrt{2} - 1\right) \notin \mathbb{Q}. \tag{11}$$

Consequently

$$\mathcal{NI}(f,[0,1]) = 1, \quad \mathcal{NI}\left(f,\left[1,\sqrt{2}\right]\right) = \mathcal{NI}\left(f,\left[0,\sqrt{2}\right]\right) = 0. \tag{12}$$

Question 4. (5 pts) *Let $a \in \mathbb{R}$. Let $f, g: \mathbb{R} \mapsto \mathbb{R}$ be such that*

i. f, g are differentiable on $\mathbb{R} - \{a\}$;

ii. $\lim_{x\to a} f(x) = +\infty$; $\lim_{x\to a} g(x) = +\infty$;

iii. $\lim_{x\to a} \frac{f'(x)}{g'(x)} = +\infty$.

Prove that $\lim_{x\to a} \frac{f(x)}{g(x)} = +\infty$.

Proof. We prove $\lim_{x\to a+} \frac{f}{g} = \lim_{x\to a-} \frac{f}{g} = +\infty$. We prove the first one here as the second one is almost identical.

Let $M > 0$ be arbitrary.

As $\lim_{x\to a+} \frac{f'(x)}{g'(x)} = +\infty$, there is $\delta_1 > 0$ such that for all $0 < x - a < \delta_1$,

$$\frac{f'(x)}{g'(x)} > 2M. \tag{13}$$

Now let x_1 satisfy $0 < x_1 - a < \delta_1$. As $\lim_{x\to a+} f(x) = +\infty$, there is $\delta_2 > 0$ such that $\delta_2 < x_1 - a$ and for all $0 < x - a < \delta_2$,

$$f(x) > 3|f(x_1)|. \tag{14}$$

Similarly there is $\delta_3 > 0$ such that $\delta_3 < x_1 - a$ and for all $0 < x - a < \delta_3$,

$$g(x) > 3|g(x_1)|. \tag{15}$$

Now let $\delta = \min\{\delta_2, \delta_3\}$. For every $0 < x - a < \delta$, we have

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(x_1)}{g(x) - g(x_1)} \cdot \frac{1 - \frac{g(x_1)}{g(x)}}{1 - \frac{f(x_1)}{f(x)}} = \frac{f'(c)}{g'(c)} \cdot \frac{1 - \frac{g(x_1)}{g(x)}}{1 - \frac{f(x_1)}{f(x)}} > 2\,M \cdot \frac{2/3}{4/3} = M. \tag{16}$$

Thus ends the proof. □