MATH 117 FALL 2014 LECTURE 42 (Nov. 20, 2014)

Read:

- Infinite Series
  - Definition.
  - Proving convergence: Definition; Cauchy; Monotone + bound;
  - If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$;
  - Comparison:
    - $|a_n| \leqslant b_n$, $\sum_{n=1}^{\infty} b_n$ converges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ converges;
    - $a_n \geqslant b_n \geqslant 0$, $\sum_{n=1}^{\infty} b_n$ diverges $\Longrightarrow \sum_{n=1}^{\infty} a_n$ diverges.
  - Typical series.
    - $\sum_{n=1}^{\infty} r^n$ converges if and only if $|r| < 1$;
    - $\sum_{n=1}^{\infty} \frac{1}{n^a}$ converges if and only if $a > 1$;
    - $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges.
- Continuity
  - Definition; ε-δ.
  - Continuity of $f \pm g, c\, f, f\, g, f\,/\,g, g \circ f$, inverse function.
  - f continuous on $[a, b]$ then f reaches maximum and minimum.
  - IVT.
- Differentiability/differentiation
  - Definition.
  - Differentiability and derivatives of $f \pm g, c\, g, f\, g, f\,/\,g, g \circ f$, inverse function.
  - MVT;
  - Use derivative to determine monotonicity and find maximizers/minimizers.
- Examples

Example 1. Prove that $\sum_{n=1}^{\infty} \sin\frac{1}{n^2}$ is convergent.

Solution. We prove $\sin\frac{1}{n^2} \leqslant \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and convergence follows. It suffices to prove $\sin x \leqslant x$ for all $x > 0$. Let $f(x) := x - \sin x$. We have $f'(x) = 1 - \cos x \geqslant 0$ for all x and therefore $f(x)$ is increasing. Now as $f(0) = 0$ we see that for every $x > 0$, $f(x) \geqslant f(0) = 0$.

Exercise 1. Prove that $\sum_{n=1}^{\infty} \sin\frac{1}{n}$ is divergent.

Example 2. Let $f: [a, b] \mapsto \mathbb{R}$ be continuous and $f(x) > 0$ for every $x \in [a, b]$. Prove there is $\delta > 0$ such that $\forall x \in [a, b]$, $f(x) > \delta$.

Proof. As f is continuous on $[a, b]$ there is a minimizer $x_0 \in [a, b]$, that is

$$\forall x \in [a, b], \qquad f(x) \geqslant f(x_0). \tag{1}$$

By assumption $f(x_0) > 0$. Taking $\delta = f(x_0)/2$ ends the proof. $\square$

Example 3. Prove that $f(x) := \tan x - x$ is invertible on $(-\pi/2, \pi/2)$.

Solution.

- Onto.

  Let $y \in \mathbb{R}$ be arbitrary. As $\lim_{x \to \pi/2} f(x) = +\infty$ there is $x_1 \in (-\pi/2, \pi/2)$ such that $f(x_1) > y$. Similarly there is $x_2 \in (-\pi/2, \pi/2)$ such that $f(x_2) < y$. By IVT there is x between x_1, x_2 such that $f(x) = y$.

- One-to-one.

  We prove f is strictly increasing. Let $x_1 < x_2$ be arbitrary from $(-\pi/2, \pi/2)$. There are three cases.

  - $x_1, x_2 \geqslant 0$.

    By MVT we have

    $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) = \frac{1}{\cos^2 c} - 1. \tag{2}$$

    for some $c \in (x_1, x_2)$. As in particular $c > 0$ we have $f'(c) > 0$ and consequently $f(x_2) > f(x_1)$.

  - $x_1, x_2 \leqslant 0$.

    Similarly we have $f(x_2) > f(x_1)$ in this case too.

  - $x_1 < 0 < x_2$.

    From the above two cases we see that $f(x_1) < 0 < f(x_2)$.