MATH 117 FALL 2014 LECTURE 40 (Nov. 17, 2014)

Read: Bowman §4.F, **314** Differentiation §3.2, §3.3.

- L'Hospital's Rule.

THEOREM 1. *Let $a \in \mathbb{R}$. If there is $\delta > 0$ such that*

i. *f, g differentiable on $(a-\delta, a+\delta) - \{a\}$;*

ii. *$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$;*

iii. *$\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ (L could be a real number of $\pm\infty$);*

iv. *$g'(x) \neq 0$ on $(a-\delta, a+\delta) - \{a\}$;*

Then $\lim_{x \to a} \frac{f(x)}{g(x)} = L$.

Remark 2. "$x \to a$" can be replaced by anyone of the following

$$x \to +\infty; \quad x \to -\infty; \quad x \to a+; \quad x \to a-. \tag{1}$$

Example 3. Calculate $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$.

Solution. Let $f(x) = 1 - \cos x$, $g(x) = x^2$. Take $\delta = 1$. Obviously i, ii, iv are satisfied. For iii we calculate $f'(x) = \sin x$, $g'(x) = 2x$. It is not clear what $\lim_{x \to 0} \frac{\sin x}{2x}$ is. However we notice that $\sin x$ and $2x$ also satisfies i, ii, iv on $(-1, 1) - \{0\}$ and therefore we could try to apply L'Hospital's rule to calculate $\lim_{x \to 0} \frac{\sin x}{2x}$. Taking derivatives we have

$$\lim_{x \to 0} \frac{(\sin x)'}{(2x)'} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}. \tag{2}$$

Therefore $\lim_{x \to 0} \frac{\sin x}{2x} = \frac{1}{2}$ and consequently $\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$.

Exercise 1. Calculate $\lim_{x \to 1} \frac{\ln x}{x - 1}$.

Exercise 2. Calculate $\lim_{x \to 0} \frac{x - \sin x}{x^2 \sin x}$.

Example 4. Calculate $\lim_{x \to 0+} \frac{\sqrt{x}}{1 - e^{2\sqrt{x}}}$.

Solution. Observe that it is equivalent to calculate $\lim_{t \to 0+} \frac{t}{1 - e^{2t}}$ (see Exercise 4). Application of L'Hospital to this limit gives the limit to be

$$\lim_{t \to 0+} \frac{t'}{(1 - e^{2t})'} = \lim_{t \to 0+} \frac{1}{-2e^{2t}} = -\frac{1}{2}. \tag{3}$$

Exercise 3. Convince yourself that direct application of L'Hospital to $\lim_{x \to 0+} \frac{\sqrt{x}}{1 - e^{2\sqrt{x}}}$ is not a good idea.

Exercise 4. Prove that

$$\lim_{x \to 0+} \frac{\sqrt{x}}{1 - e^{2\sqrt{x}}} = L \Longleftrightarrow \lim_{t \to 0+} \frac{t}{1 - e^{2t}} = L. \tag{4}$$

Exercise 5. Show that L'Hospital's rule in general does not hold if assumption ii is dropped.

- Proof of L'Hospital.

THEOREM 5. (CAUCHY'S MVT) *Let $f(x), g(x): [a, b] \mapsto \mathbb{R}$ satisfy*

i. *f, g are differentiable on (a, b);*

ii. f, g *are continuous on* $[a, b]$*;*

iii. $g'(x) \neq 0$ *on* (a, b)*.*

Then there is $c \in (a, b)$ *such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}. \tag{5}$$

Proof. Define

$$h(x) := f(x)\,(g(b) - g(a)) - g(x)\,(f(b) - f(a)). \tag{6}$$

Then one easily checks that h satisfies the three conditions for Rolle's Theorem. Thus there is $c \in (a, b)$ such that $h'(c) = 0$ and the conclusion follows. □

Proof. (OF L'HOSPITAL'S RULE) We prove it for the case $L \in \mathbb{R}$. Define

$$F(x) := \begin{cases} f(x) & x \neq a \\ 0 & x = a \end{cases}; \qquad G(x) := \begin{cases} g(x) & x \neq a \\ 0 & x = a \end{cases}. \tag{7}$$

Then F, G are continuous on $(a - \delta, a + \delta)$ (Exercise 6).

Let $\varepsilon > 0$ be arbitrary. As $\lim_{x \to a} \frac{f'(x)}{g'(x)} = L$ there is $\delta_1 > 0$ such that

$$\forall 0 < |x - a| < \delta_1, \qquad \left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon. \tag{8}$$

Now consider an arbitrary x such that $0 < |x - a| < \min \{\delta_1, \delta\}$. We have

$$\frac{f(x)}{g(x)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F'(c)}{G'(c)} = \frac{f'(c)}{g'(c)} \tag{9}$$

for some c between x and a. For such c we have $0 < |c - a| < \delta_1$ and thus

$$\left| \frac{f(x)}{g(x)} - L \right| = \left| \frac{f'(c)}{g'(c)} - L \right| < \varepsilon. \tag{10}$$

Thus by definition $\lim_{x \to a} \frac{f(x)}{g(x)} = L$. □

Exercise 6. Prove that F, G are continuous on $(a - \delta, a + \delta)$.

Exercise 7. Prove L'Hospital's Rule for the cases $L = \pm\infty$.

- L'Hospital's Rule in other situations.
  - We face the following situations when studying limits:

$$\lim\,(c\,f), \quad \lim(f + g), \quad \lim(f\,g), \quad \lim \frac{f}{g}. \tag{11}$$

When we allow $\lim f$ and $\lim g$ to take $\pm\infty$ (and 0 in the last situation), we find ourselves facing several "undetermined" cases. For example

$$\lim f = +\infty, \lim g = -\infty, \qquad \lim\,(f + g) = ? \tag{12}$$

Exercise 8. Show that $\lim\,(f + g)$ could be any one of the four possibilities: a real number, $\pm\infty$, does not exist.

The situation in (12) is oftened denoted as $\infty - \infty$. The case we have just settled above is $\frac{0}{0}$.

It turns out that all the "undetermined" cases could be reduced to either $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

- Naively, one may think $\frac{\infty}{\infty}$ could further be reduced to $\frac{0}{0}$ through setting $F = \frac{1}{f}$ and $G = \frac{1}{g}$. However this is not practical.

  Example 6. Consider $\lim_{x \to \infty} x^3\, e^{-x}$. If we let $f(x) := e^{-x}$ and $g(x) := x^{-3}$, then we have a $\frac{0}{0}$ type limit $\lim_{x \to \infty} \frac{f(x)}{g(x)}$. However a bit calculation would reveal that application of L'Hospital to this pair of f, g does not lead us anywhere.

- It turns out that there is a L'Hospital's rule for $\frac{\infty}{\infty}$ where the only major difference from Theorem 1 is that ii becomes $\lim_{x \to a} f = +\infty$ (or $-\infty$), $\lim_{x \to a} g = +\infty$ (or $-\infty$).

  Exercise 9. It is tempting to prove L'Hospital's Rule for $\frac{\infty}{\infty}$ as follows:
  Let $F := 1/f$, $G := 1/g$ and apply L'Hospital's Rule for $\frac{0}{0}$.
  Explain why the situation is not so simple.