Math 117 Fall 2014 Lecture 39 (Nov. 14, 2014)

Read: Bowman §5.E, **314** Integration §3.

- Evaluation of Integrals. So far we have
  - By defintion.
    1. Let P be an arbitrary partition of $[a, b]$. Calculate $U(f, P)$, $L(f, P)$ and simplify if possible.
    2. Calculate $U(f) := \inf_P U(f, P)$, $L(f) := \sup_P L(f, P)$.
    3. If $U(f) = L(f)$ then f is integrable on $[a, b]$ with $\int_a^b f(x)\,\mathrm{d}x = U(f) = L(f)$. If $U(f) \neq L(f)$ then f is not integrable on $[a, b]$.
  - By clever choice of partitions.
    Find a sequence of partitions P_n such that $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) \in \mathbb{R}$.
- Fundamental Theorems of Calculus

THEOREM 1. (FTC VERSION 1) *Let $f: [a, b] \mapsto \mathbb{R}$ and $F: [a, b] \mapsto \mathbb{R}$ satisfy*

i. f is integrable on $[a, b]$;

ii. F is differentiable on (a, b) with $F' = f$ on (a, b);

iii. F is continuous on $[a, b]$.

Then we have

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a). \tag{1}$$

Proof. Let P be an arbitrary partition of $[a, b]$, $P: a = x_0 < x_1 < \cdots < x_n = b$. Then for every $k \in \{1, \ldots, n\}$ we see that $F(x)$ satisfies the conditions for MVT. Therefore

$$\begin{aligned} F(b) - F(a) &= [F(x_n) - F(x_{n-1})] + [F(x_{n-1}) - F(x_{n-2})] + \cdots + [F(x_1) - F(x_0)] \\ &= f(c_n)\,(x_n - x_{n-1}) + \cdots + f(c_1)\,(x_1 - x_0) \\ &\leqslant \left(\sup_{[x_{n-1}, x_n]} f\right)(x_n - x_{n-1}) + \cdots + \left(\sup_{[x_0, x_1]} f\right)(x_1 - x_0) \\ &= U(f, P). \end{aligned} \tag{2}$$

Here the inequality is because $c_k \in (x_{k-1}, x_k)$ and it then follows $f(c_k) \leqslant \sup_{[x_{k-1}, x_k]} f$. Thus we have shown $F(b) - F(a) \leqslant U(f, P)$ for every partition P. Taking infimum of both sides we have $F(b) - F(a) \leqslant U(f)$.

Similarly we can prove $F(b) - F(a) \geqslant L(f)$. Since f is integrable, $U(f) = L(f) = \int_a^b f(x)\,\mathrm{d}x$ and the conclusion follows. □

Example 2. Evaluate $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$.

Solution. We know that

$$(\arctan x)' = \frac{1}{1+x^2} \tag{3}$$

for all $x \in \mathbb{R}$. Therefore the conditions for FTCV1 are all satisfied. Consequently

$$\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x = \arctan 1 - \arctan 0 = \frac{\pi}{4}. \tag{4}$$

Exercise 1. Try to evaluate $\int_0^1 \frac{1}{1+x^2}\,\mathrm{d}x$ through definition or through clever choice of partitions, and appreciate the power of FTCV1.

Exercise 2. Prove that there is no $F\colon [0,1] \mapsto \mathbb{R}$ such that on $(0,1)$, $F'(x) = R(x) := \begin{cases} \frac{1}{q} & x = \frac{p}{q} \\ 0 & x \notin \mathbb{Q} \end{cases}$, although the Riemann function $R(x)$ is integrable on $[0,1]$.

THEOREM 3. (FTC VERSION 2) *Let $f\colon [a,b] \mapsto \mathbb{R}$ be integrable on $[a,b]$.*

a) *then $G(x) := \int_a^x f(t)\,\mathrm{d}t$ is defined for every $x \in [a,b]$ and furthermore is continuous on $[a,b]$.*
b) *if furthermore f is continuous at $c \in (a,b)$, $G(x)$ is differentiable at c with $G'(c) = f(c)$.*

The proof of Theorem 3 relies on the following lemmas.

LEMMA 4. *Let $f(x)$ be integrable on $[a,b]$. Then $f(x)$ is bounded on $[a,b]$.*

Proof. Assume the contrary. Then there is either $\{c_n\} \subset [a,b]$ such that $\lim_{n\to\infty} f(c_n) = +\infty$ or $\{c_n\} \subset [a,b]$ such that $\lim_{n\to\infty} f(c_n) = -\infty$. Wlog assume the former is true.

Let P be an arbitrary partition of $[a,b]$, $P\colon a = x_0 < x_1 < \cdots < x_m = b$. Then there is $k_0 \in \{1,2,\ldots,n\}$ such that $[x_{k_0-1}, x_{k_0}]$ contains infinitely many c_n's and consequently

$$\sup_{[x_{k_0-1}, x_{k_0}]} f = +\infty. \tag{5}$$

Now we have

$$U(f,P) = \sum_{k=1}^{n} \left(\sup_{[x_{k-1}, x_k]} f \right) (x_k - x_{k-1}) \geqslant \sum_{k \neq k_0} f(x_k)\,(x_k - x_{k-1}) + \infty = +\infty. \tag{6}$$

Therefore $U(f) = \inf_P U(f,P)$ cannot be finite and f cannot be integrable. □

Exercise 3. Prove Lemma 4 directly as follows.

i. Prove that if f is integrable on $[a,b]$, so is $|f|$.
ii. Let $L := \int_a^b |f(x)|\,\mathrm{d}x$. There is P such that $U(|f|, P) \leqslant L + 1$. Now prove that $\sup_{[x_{k-1},x_k]} f < \infty$ for each k.

LEMMA 5. *Let f, g, h be integrable on $[a,b]$ and $\forall x \in [a,b]$, $f(x) \leqslant g(x) \leqslant h(x)$. Then*

$$\int_a^b f(x)\,\mathrm{d}x \leqslant \int_a^b g(x)\,\mathrm{d}x \leqslant \int_a^b h(x)\,\mathrm{d}x. \tag{7}$$

Exercise 4. Prove Lemma 5.

Exercise 5. Prove or disprive: Let f, g, h be integrable on $[a,b]$ and $\forall x \in [a,b]$, $f(x) < g(x) < h(x)$. Then

$$\int_a^b f(x)\,\mathrm{d}x < \int_a^b g(x)\,\mathrm{d}x < \int_a^b h(x)\,\mathrm{d}x. \tag{8}$$

What if we further assume f, g, h are all continuous?

Exercise 6. Let f be integrable on $[a,b]$. Prove that $\left| \int_a^b f(x)\,\mathrm{d}x \right| \leqslant \int_a^b |f(x)|\,\mathrm{d}x$.

Proof. (FTC VERSION 2)

a) Since f is integrable on $[a,b]$ it is also integrable on every $[a,x]$ so G is well-defined for all $x \in [a,b]$. As f is bounded, there is $M > 0$ such that $-M < f(x) < M$ for all $x \in [a,b]$.
   - Right continuity at a.

For every $x > a$ we have

$$G(x) - G(a) = \int_a^x f(t)\,\mathrm{d}t \leqslant \int_a^x M\,\mathrm{d}x = M\,(x - a). \tag{9}$$

Similarly

$$G(x) - G(a) \geqslant -M\,(x - a). \tag{10}$$

By Squeeze Theorem we conclude $\lim_{x \to a+} G(x) = G(a)$.

- Continuity at $c \in (a, b)$ and left continuity at b: Exercises.

b) Let $c \in (a, b)$ be arbitrary. We prove $\lim_{x \to c+} \frac{G(x) - G(c)}{x - c} = f(c)$ and left $\lim_{x \to c-} \frac{G(x) - G(c)}{x - c} = f(c)$ as an exercise.

Let $\varepsilon > 0$ be arbitrary. As f is continuous at c, there is $\delta > 0$ such that when $|x - c| < \delta$, $|f(x) - f(c)| < \varepsilon$. Now for every $0 < x - c < \delta$ we have

$$\begin{aligned} \left| \frac{G(x) - G(c)}{x - c} - f(c) \right| &= \frac{1}{x - c} |[G(x) - G(c)] - f(c)\,(x - c)| \\ &= \frac{1}{x - c} \left| \int_c^x f(t)\,\mathrm{d}t - \int_c^x f(c)\,\mathrm{d}t \right| \\ &= \frac{1}{x - c} \left| \int_c^x [f(t) - f(c)]\,\mathrm{d}x \right| \\ &\leqslant \frac{1}{x - c} \int_c^x |f(t) - f(c)|\,\mathrm{d}x \\ &\leqslant \frac{1}{x - c} \int_c^x \varepsilon\,\mathrm{d}x = \varepsilon. \end{aligned} \tag{11}$$

Thus ends the proof. □

Example 6. Let $G(x) := \int_1^x e^{-t^2}\,\mathrm{d}t$, $G_1(x) := \int_1^{x^3} e^{-t^2}\,\mathrm{d}t$, $G_2(x) := \int_{\sin x}^{x^3} e^{-t^2}\,\mathrm{d}t$. Prove that $G(x), G_1(x), G_2(x)$ are differentiable on $\mathbb{R}$ and calculate their derivatives.

Solution. We know that e^{-t^2} is continuous on $\mathbb{R}$ and integrable on every $[1, x]$, therefore $G(x)$ is differentiable on $\mathbb{R}$. The differentiability of $G_1(x)$ and $G_2(x)$ follows from

$$G_1(x) = G(x^3), \qquad G_2(x) = G(x^3) - G(\sin x). \tag{12}$$

Now we easily calculate

$$G'(x) = e^{-x^2}; \quad G_1'(x) = 3\,x^2\,e^{-x^6}; \quad G_2'(x) = 3\,x^2\,e^{-x^6} - e^{-(\sin x)^2} \cos x. \tag{13}$$