MATH 117 FALL 2014 LECTURE 38 (Nov. 13, 2014)

Read: Bowman §5.B – 5.D, **314** Integration §2.

- Integrability by definition. $f: [a, b] \mapsto \mathbb{R}$.

  1. Let P be an arbitrary partition of $[a, b]$. Calculate $U(f, P), L(f, P)$ and simplify if possible.
  2. Calculate $U(f) := \inf_P U(f, P)$, $L(f) := \sup_P L(f, P)$.
  3. If $U(f) = L(f)$ then f is integrable on $[a, b]$ with $\int_a^b f(x)\,\mathrm{d}x = U(f) = L(f)$. If $U(f) \neq L(f)$ then f is not integrable on $[a, b]$.

  Exercise 1. Prove the integrability of x on $[0, 1]$ by definition. (Hint:[1])

- Integrability criteria.

  THEOREM 1. *Let $f: [a, b] \mapsto \mathbb{R}$. Then f is integrable on $[a, b]$ if and only if there is a sequence of partitions P_n of $[a, b]$ and $L \in \mathbb{R}$ such that $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = L$.*

  Proof. We leave the "only if" part as exercise.

  - "If".
    - $U(f) \leqslant L(f)$.

      We have $U(f) = \inf_P U(f, P) \leqslant U(f, P_n)$ for every $n \in \mathbb{N}$. Thus by Comparison Theorem $U(f) \leqslant L = \lim_{n\to\infty} U(f, P_n)$. Similarly we have $L(f) \geqslant L$. Therefore $U(f) \leqslant L(f)$.
    - $U(f) \geqslant L(f)$.

      First note that for every partition P of $[a, b]$, we have

      $$U(f, P) = \sum_{k=1}^{n} \left(\sup_{[x_{k-1}, x_k]} f \right) (x_k - x_{k-1}) \geqslant \sum_{k=1}^{n} \left(\inf_{[x_{k-1}, x_k]} f \right) (x_k - x_{k-1}) = L(f, P). \tag{1}$$

      Let P, Q be two arbitrary partitions of $[a, b]$. Then $P \cup Q$ refines both P and Q and consequently

      $$U(f, P) \geqslant U(f, P \cup Q) \geqslant L(f, P \cup Q) \geqslant L(f, Q). \tag{2}$$

      This gives

      $$U(f) = \inf_P U(f, P) \geqslant \inf_P L(f, Q) = L(f, Q) \tag{3}$$

      and furthermore

      $$L(f) = \sup_Q L(f, Q) \leqslant \sup_Q U(f) = U(f). \tag{4}$$

    Since $U(f) \leqslant L(f)$ and $U(f) \geqslant L(f)$ hold at the same time, there must hold $U(f) = L(f)$. □

  Example 2. Prove the integrability of $f(x) = x$ on $[0, 1]$.

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1. Let $a_k := x_k - x_{k-1}$. Then we have $\sum_{k=1}^n a_k = 1$ and try to minimize $U(f, P) = \sum_{1 \leqslant i \leqslant j \leqslant n} a_i\, a_j$. Study $2\,U(f, P) - (\sum_{k=1}^n a_k)^2$.

Solution. Take $P_n = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}, 1\right\}$. Then we have

$$U(f, P_n) = \sum_{k=1}^{n} \frac{k}{n} \left(\frac{k}{n} - \frac{k-1}{n}\right) = \frac{1}{n^2} \sum_{k=1}^{n} k = \frac{n+1}{2\,n} \tag{5}$$

and similarly $L(f, P_n) = \frac{n-1}{2\,n}$. Clearly $\lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} L(f, P_n) = \frac{1}{2}$ and the proof ends.

Exercise 2. Prove the integrability of $f(x) = x^2$ on $[0, 1]$.

Exercise 3. Prove the integrability of $f(x) = x^3$ on $[0, 1]$.

Exercise 4. Let $f: [a, b] \mapsto \mathbb{R}$. Prove or disprove: f is integrable on $[a, b]$ if and only if there is a sequence of partitions P_n such that $\lim_{n\to\infty} [U(f, P_n) - L(f, P_n)] = 0$.

Problem 1. Let $f: [a, b] \mapsto \mathbb{R}$ be monotone. Prove that f is integrable on $[a, b]$.

Problem 2. Let $f: [a, b] \mapsto \mathbb{R}$ be continuous. Prove that f is integrable on $[a, b]$. (Hint: f is uniformly continuous). Give an example of a discontinuous monotone function.

- Properties.

LEMMA 3. *Let f be integrable on $[a, b]$. Then f is bounded on $[a, b]$.*

Proof. Exercise. □

THEOREM 4. *Let f, g be integrable on $[a, b]$. Let $c \in \mathbb{R}$. Then*

a) $c f$ is integrable on $[a, b]$ with $\int_a^b (c f)(x)\,\mathrm{d}x = c \int_a^b f(x)\,\mathrm{d}x$;

b) $f \pm g$ is integrable on $[a, b]$ with $\int_a^b (f \pm g)(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x \pm \int_a^b g(x)\,\mathrm{d}x$;

c) $f g$ is integrable on $[a, b]$.

d) If there is $c_0 > 0$ such that $|g(x)| > c_0$, then $\frac{f}{g}$ is integrable on $[a, b]$.

Proof. Exercises. (See 314 Integration §2 for b) and c)). □

Exercise 5. Find two functions f, g that are integrable on $[0, 1]$, $\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x = 0$, while $\int_0^1 (f g)(x)\,\mathrm{d}x = 1$.

Exercise 6. Find two functions f, g that are integrable on $[0, 1]$, $\int_0^1 f(x)\,\mathrm{d}x = \int_0^1 g(x)\,\mathrm{d}x = 1$, while $\int_0^1 (f g)(x)\,\mathrm{d}x = 0$.

Problem 3. Let $a, b, c \in \mathbb{R}$ be arbitrary. Prove or disprove: There are functions f, g integrable on $[0, 1]$ such that $\int_0^1 f(x)\,\mathrm{d}x = a$, $\int_0^1 g(x)\,\mathrm{d}x = b$, $\int_0^1 (f g)(x)\,\mathrm{d}x = c$.

THEOREM 5. *Let $f: [a, b] \mapsto \mathbb{R}$ and let $c \in (a, b)$. Then f is integrable on $[a, b]$ if and only if f is integrable on both $[a, c]$ and $[c, b]$. Furthermore*

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x. \tag{6}$$

Proof. We prove "only if". Let f be integrable on $[a, b]$ and we will prove f is integrable on $[a, c]$. The integrability of f on $[c, b]$ can be proved almost identically.

Let $g(x) := \begin{cases} 1 & x \in [a, c] \\ 0 & x \notin [a, c] \end{cases}$. Clearly $g(x)$ is integrable on $[a, b]$. Thus by Theorem 4 the function $\tilde{f}(x) := f(x)\, g(x) = \begin{cases} f(x) & x \in [a, c] \\ 0 & x \in [c, b] \end{cases}$ is integrable on $[a, b]$. Thus there is a sequence of partitions of $[a, b]$, denoted P_n, such that $\lim_{n \to \infty} [U(f, P_n) - L(f, P_n)] = 0$. Now set $Q_n := P_n \cap [a, c] \cup \{c\}$. Then every Q_n is a partition of $[a, c]$ and

$$0 \leqslant U(f, Q_n) - L(f, Q_n) = U(f, P_n \cup \{c\}) - L(f, P_n \cup \{c\}) \leqslant U(f, P_n) - L(f, P_n). \tag{7}$$

Now application of Squeeze Theorem gives $\lim_{n \to \infty} [U(f, Q_n) - L(f, Q_n)] = 0$ and integrability follows. □

Exercise 7. Prove the "if" part of Theorem 5.

Exercise 8. Prove the "furthermore" part of Theorem 5.

DEFINITION 6. *Let $a, b \in \mathbb{R}$ with $a > b$. We say f is integrable on $[a, b]$ if and only if f is integrable on $[b, a]$ and define*

$$\int_a^b f(x)\, \mathrm{d}x := -\int_b^a f(x)\, \mathrm{d}x. \tag{8}$$

Exercise 9. Let $a, b, c \in \mathbb{R}$ be arbitrary, and let f be integrable on $[\min\{a, b, c\}, \max\{a, b, c\}]$. Then there holds

$$\int_a^b f(x)\, \mathrm{d}x = \int_a^c f(x)\, \mathrm{d}x + \int_c^b f(x)\, \mathrm{d}x. \tag{9}$$

- Integrability of the Riemann function $R(x) := \begin{cases} \frac{1}{q} & x = \frac{p}{q} \text{ with } (p, q) = 1, q > 0 \\ 0 & x \notin \mathbb{Q} \end{cases}$ on $[0, 1]$.

Proof. It is easy to prove that $L(R, P) = 0$ for every partition P of $[0, 1]$ (exercise). Let $\varepsilon > 0$ be arbitrary. We will construct a partition P_ε such that $U(R, P_\varepsilon) < \varepsilon$.

First note that there are only finitely many x such that $R(x) > \frac{\varepsilon}{3}$. Denote them by $c_1 < c_2 < \cdots < c_K$. Let $\delta_1 := \min(c_k - c_{k-1})$. Now set $\delta := \min\left\{\frac{\varepsilon}{6K}, \frac{\delta_1}{3}, \frac{c_1}{3}, \frac{1 - c_K}{3}\right\}$. Define

$$P_\varepsilon := \{0, c_1 - \delta, c_1 + \delta, c_2 - \delta, c_2 + \delta, \ldots, c_K - \delta, c_K + \delta, 1\}. \tag{10}$$

Then as $R(x) \leqslant 1$ for all x, we have

$$\sup_{[c_k - \delta, c_k + \delta]} R(x) \leqslant 1. \tag{11}$$

On the other hand on other sub-intervals the supreme of $R(x) \leqslant \frac{\varepsilon}{3}$. Therefore

$$U(R, P_\varepsilon) \leqslant 2\, \delta\, K + \frac{\varepsilon}{3} \leqslant \frac{2\, \varepsilon}{3} < \varepsilon. \tag{12}$$

Thus ends the proof. □