# Math 117 Fall 2014 Lecture 37 (Nov. 12, 2014)

**Read: Bowman** §5.A, **314** Integration §1.

- Riemann Integration.
  - We define Riemann integral as follows.

    Definition 1. (Partition of an inverval) *Let $a, b \in \mathbb{R}, a < b$. A partition of $[a, b]$ is a set $P \subseteq [a, b]$ such that i. $P$ is finite, ii. $a, b \in P$.*

    **Example 2.** $\{0, 1\}$, $\left\{0, \frac{1}{2}, \frac{2}{3}, 1\right\}$ are partitions of $[0, 1]$; $\left\{\frac{1}{n} | \, n \in \mathbb{N}\right\} \cup \{0\}$, $\{0, 1, 2\}$, $\left\{0, \frac{1}{2}, \frac{2}{3}\right\}$ are not partitions of $[0, 1]$.

    Definition 3. (Upper/Lower sum) *Let $f: [a, b] \mapsto \mathbb{R}$. Let $P$ be a partition of $[a, b]$, denoted as $P = \{x_0, \ldots, x_n\}$ with $a = x_0 < x_1 < \cdots < x_n = b$. Then we define the upper/lower sums as:*

    $$U(f, P) := \sum_{k=1}^{n} M_k \, (x_k - x_{k-1}); \qquad L(f, P) := \sum_{k=1}^{n} m_k \, (x_k - x_{k-1}) \tag{1}$$

    *where*

    $$M_k := \sup_{[x_{k-1}, x_k]} f(x), \qquad m_k := \inf_{[x_{k-1}, x_k]} f(x). \tag{2}$$

    Definition 4. (Upper/Lower integrals) *Let $f: [a, b] \mapsto \mathbb{R}$. Define its upper/lower integrals as*

    $$U(f) := \inf_{P \text{ is a partition for } [a,b]} U(f, P); \quad L(f) := \sup_{P \text{ is a partition for } [a,b]} L(f, P). \tag{3}$$

    Definition 5. (Riemann Integrability) *Let $f: [a, b] \mapsto \mathbb{R}$. Then $f$ is Riemann integrable on $[a, b]$ if and only if $U(f) = L(f) \in \mathbb{R}$. In this case the common value is called the Riemann integral of $f$ over $[a, b]$ (or from $a$ to $b$) and denoted $\int_a^b f(x) \, \mathrm{d}x$.*

  - Examples.

    **Example 6.** Prove that $f(x) = 1$ is integrable on $[0, 1]$ and find the integral.

    **Proof.** Let $P = \{x_0, x_1, \ldots, x_n\}$ be an arbitrary partition with $a = x_0 < x_1 < \cdots < x_n = b$. Then for every $k \in \{1, 2, \ldots, n\}$ we have

    $$\sup_{[x_{k-1}, x_k]} f(x) = \inf_{[x_{k-1}, x_k]} f(x) = 1. \tag{4}$$

    Thus

    $$U(f, P) = 1; \qquad L(f, P) = 1. \tag{5}$$

    As this holds for every partition $P$, we further have

    $$U(f) = 1 = L(f). \tag{6}$$

    Thus by definition $f$ is integrable with $\int_0^1 f(x) \, \mathrm{d}x = 1$. □

    **Example 7.** Prove that $D(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is not integrable on $[0, 1]$.

**Proof.** Let $P=\{x_0,x_1,\ldots,x_n\}$ be an arbitrary partition with $a=x_0<x_1<\cdots<x_n=b$. Then for every $k\in\{1,2,\ldots,n\}$ we have

$$\sup_{[x_{k-1},x_k]} f(x)=1; \qquad \inf_{[x_{k-1},x_k]} f(x)=0. \tag{7}$$

Consequently

$$U(f,P)=1, \qquad L(f,P)=0. \tag{8}$$

As this holds for every partition $P$, we conclude

$$U(f)=1, \qquad L(f)=0. \tag{9}$$

Since $1\neq 0$ $D(x)$ is not integrable on $[0,1]$. □

**Exercise 1.** Prove by definition the integrability of $f(x)=\begin{cases}1 & x>0\\ 0 & x\leqslant 0\end{cases}$ on $[-1,1]$ and find $\int_{-1}^{1} f(x)\,\mathrm{d}x$.

- Refinement of partition.

DEFINITION 8. *Let $P,Q$ be partitions of $[a,b]$. Say $Q$ refines $P$ if and only if $P\subseteq Q$.*

**Example 9.** $\{0,1\},\left\{0,\frac{1}{2},\frac{2}{3},1\right\}$ are both partitions of $[0,1]$ and the latter refines the former.

LEMMA 10. *Let $f\colon[a,b]\mapsto\mathbb{R}$ and $P,Q$ be partitions of $[a,b]$ with $P\subseteq Q$. Then*

$$U(f,P)\geqslant U(f,Q); \qquad L(f,P)\leqslant L(f,Q). \tag{10}$$

**Proof.** We prove the first one and leave the second one as exercise.

Denote $P=\{x_0,\ldots,x_n\}$ and $Q=\{y_0,\ldots,y_m\}$. As $P\subseteq Q$, $Q-P$ consists of $k=m-n$ elements, we denote them by $z_1,\ldots z_k$. Now define

$$Q_1=P\cup\{z_1\},\ Q_2=P\cup\{z_1,z_2\},\ \ldots,Q_{k-1}=P\cup\{z_1,\ldots,z_{k-1}\}. \tag{11}$$

It suffices to prove

$$U(f,P)\geqslant U(f,Q_1)\geqslant U(f,Q_2)\geqslant\cdots\geqslant U(f,Q_{k-1})\geqslant U(f,Q). \tag{12}$$

It is clear now that it suffices to prove the following: Let $P=\{x_0,\ldots,x_n\}$ be an arbitrary partition of $[a,b]$. Let $\tilde{x}\in[a,b]$ be different from $x_0,\ldots,x_n$, then

$$U(f,P)\geqslant U(f,P\cup\{\tilde{x}\}). \tag{13}$$

Let $l\in\{0,\ldots,n-1\}$ be such that $\tilde{x}\in(x_l,x_{l+1})$. Then as

$$\sup_{[x_l,x_{l+1}]} f(x)\geqslant \sup_{[x_l,\tilde{x}]} f(x); \qquad \sup_{[x_l,x_{l+1}]} f(x)\geqslant \sup_{[\tilde{x},x_{l+1}]} f(x) \tag{14}$$

we have

$$\begin{aligned} U(f,P)-U(f,P\cup\{\tilde{x}\}) &= \left(\sup_{[x_l,x_{l+1}]} f(x)\right)\cdot(x_{l+1}-x_l)\\ &\quad -\left(\sup_{[x_l,\tilde{x}]} f(x)\right)\cdot(\tilde{x}-x_l)\\ &\quad -\left(\sup_{[\tilde{x},x_{l+1}]} f(x)\right)\cdot(x_{l+1}-\tilde{x})\\ &\geqslant \left(\sup_{[x_l,x_{l+1}]} f(x)\right)\cdot[(x_{l+1}-x_l)-(\tilde{x}-x_l)-(x_{l+1}-\tilde{x})]\\ &= 0. \end{aligned} \tag{15}$$

Thus ends the proof. □