Math 117 Fall 2014 Homework 8

Due Thursday Nov. 13 3pm in Assignment Box

Question 1. (5 pts) *Calculate $f'(x)$ for the following functions.*

a) (1 pt) $f_1(x) := \sqrt{\frac{x^2+1}{x^4+1}}$;

b) (1 pt) $f_2(x) := \arctan(\cos x)$.

c) (3 pts) $f_3(x) := \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leqslant 0 \end{cases}$.

Solution.

a) We have

$$f_1'(x) = \frac{1}{2}\left(\sqrt{\frac{x^2+1}{x^4+1}}\right)^{-1}\left(\frac{x^2+1}{x^4+1}\right)' = -\sqrt{\frac{x^4+1}{x^2+1}}\,\frac{x\,(x^4+2\,x^2-1)}{(x^4+1)^2}. \tag{1}$$

b) We have

$$f_2'(x) = \frac{(\cos x)'}{1+(\cos x)^2} = -\frac{\sin x}{1+(\cos x)^2}. \tag{2}$$

c) For $x < 0$, clearly $f_3'(x) = 0$. For $x > 0$ we have

$$f_3'(x) = \left(e^{-1/x}\right)' = x^{-2}\,e^{-1/x}. \tag{3}$$

At 0 we have

$$\lim_{x \to 0-} \frac{f_3(x) - f_3(0)}{x - 0} = \lim_{x \to 0-} 0 = 0; \tag{4}$$

$$\lim_{x \to 0+} \frac{f_3(x) - f_3(0)}{x - 0} = \lim_{x \to 0+} x^{-1}\,e^{-1/x}. \tag{5}$$

Now define for $x > 0$

$$g(x) := (n+1)\,2^{-n} \quad \text{for} \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right]. \tag{6}$$

Clearly we have

$$\forall x > 0, \qquad 0 \leqslant x^{-1}\,e^{-1/x} \leqslant g(x). \tag{7}$$

Now we prove $\lim_{x \to 0+} g(x) = 0$. Let $\varepsilon > 0$ be arbitrary. As $(n+1)\,2^{-n} = \frac{n+1}{(1+1)^n} < \frac{n+1}{\binom{n}{2}} = \frac{2\,(n+1)}{n\,(n-1)}$ we see that $\lim_{n \to \infty} (n+1)\,2^{-n} = 0$. Thus there is $N \in \mathbb{N}$ such that $\forall n \geqslant N$, $(n+1)\,2^{-n} < \varepsilon$. Now set $\delta = 1/N$. For every $0 < x < \delta$, we have $x < \frac{1}{N}$ which means $g(x) = (n+1)\,2^{-n}$ for some $n \geqslant N$. Consequently $|g(x)| < \varepsilon$.

Remark. Those who gave detailed proof of $\lim_{x \to 0+} \frac{f_3(x) - f_3(0)}{x - 0} = 0$ should receive one extra point.

Application of Squeeze Theorem gives $\lim_{x \to 0+} x^{-1}\,e^{-1/x} = 0$ and consequently $f_3'(0) = 0$. In summary,

$$f_3'(x) = \begin{cases} x^{-2}\,e^{-1/x} & x > 0 \\ 0 & x \leqslant 0 \end{cases}. \tag{8}$$

QUESTION 2. (5 PTS) *Find all $k \in \mathbb{Z}$ such that $|x|^k$ is differentiable everywhere on $\mathbb{R}$. Justify your claim.*

Solution. We claim that $|x|^k$ is differentiable everywhere on $\mathbb{R}$ if and only if $k \geqslant 2$ or $k = 0$.

- $|x|^k$ is not differentiable everywhere on $\mathbb{R}$ if $k < 0$. This is obvious as when $k < 0$ the function is not even defined at $x = 0$ and thus cannot be differentiable there.
- $|x|^k$ is differentiable everywhere on $\mathbb{R}$ if $k = 0$. When $k = 0$ we have $|x|^k = 1$ for all $x \in \mathbb{R}$ and differentibility follows.
- $|x|^k$ is not differentiable at 0 if $k = 1$. When $k = 1$ we have $|x|^k = \begin{cases} x & x > 0 \\ -x & x \leqslant 0 \end{cases}$. At $a = 0$ we have
$$\frac{f(x) - f(a)}{x - a} = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}. \tag{9}$$
Thus $\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = 1 \neq -1 = \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0}$ and it follows that $\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist. Consequently $|x|$ is not differentiable at $x = 0$.
- $|x|^k$ is differentiable everywhere on $\mathbb{R}$ for $k \geqslant 2$ even. In this case we have $|x|^k = x^k$ for all $x \in \mathbb{R}$ and is differentiable everywhere.
- $|x|^k$ is differentiable everywhere on $\mathbb{R}$ for $k \geqslant 2$ odd. In this case we have $|x|^k = \begin{cases} x^k & x > 0 \\ 0 & x = 0 \\ -x^k & x < 0 \end{cases} = \text{Sign}(x)\, x^k$ where the Signum function $\text{Sign}(x) := \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$. It is clear that $\text{Sign}(x)$ is differentiable at all $x \neq 0$. Consequently $|x|^k$ is differentiable at every $x \neq 0$. At $a = 0$ we check
$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0+} x^{k-1} = 0; \quad \lim_{x \to 0-} \frac{f(x) - f(0)}{x} = \lim_{x \to 0-} (-x^{k-1}) = 0. \tag{10}$$
Therefore $|x|^k$ is also differentiable at 0 with derivative 0.

QUESTION 3. (5 PTS) *Let $f(x) = 3\,x - \sin x$.*

a) (1 PT) *Prove that $f\colon \mathbb{R} \mapsto \mathbb{R}$ is one-to-one;*

b) (2 PTS) *Prove that $f\colon \mathbb{R} \mapsto \mathbb{R}$ is onto.*

c) (2 PTS) *Let $g\colon \mathbb{R} \mapsto \mathbb{R}$ be the inverse function of f, calculate $g'(0)$.*

Proof. First clearly $f(x)$ is differentiable everywhere on $\mathbb{R}$ and therefore is continuous everywhere on $\mathbb{R}$.

a) Let $x, y \in \mathbb{R}$ with $x < y$. By MVT we have there is $c \in (x, y)$ such that
$$|f(x) - f(y)| = |f'(c)\,(x - y)| = |f'(c)|\,|x - y| = |3 - \cos c|\,|x - y| \geqslant 2\,|x - y| > 0. \tag{11}$$
Therefore f is one-to-one.

b) Let $s \in \mathbb{R}$ be arbitrary. Then there are $a, b \in \mathbb{R}$ such that $3\,a - 1 > s > 3\,b + 1$. Now we have
$$f(a) = 3\,a - \sin a \geqslant 3\,a - 1 > s; \quad f(b) = 3\,b - \sin b < 3\,b + 1 < s. \tag{12}$$
By IVT there is c between a, b such that $f(c) = s$. So f is onto.

c) We have $g'(0) = \frac{1}{f'(x_0)}$ where $f(x_0) = 0$. We notice that $f(0) = 0$ so $x_0 = 0$ since f is one-to-one. Consequently $g'(0) = \frac{1}{f'(0)} = \frac{1}{2}$.

□

QUESTION 4. (5 PTS) *Find a bounded function $f(x)$ which is differentiable everywhere on $\mathbb{R}$ yet $f'(x)$ is unbounded on $\mathbb{R}$. Justify your claim.*

Solution. Let $f(x) = \sin(e^x)$. Then we have $|f(x)| \leqslant 1$ so $f(x)$ is bounded on $\mathbb{R}$. As $f(x)$ is the composition of two everywhere differentiable functions $\sin x$ and e^x, $f(x)$ is differentiable everywhere on $\mathbb{R}$. Finally, we calculate $f'(x) = e^x \cos(e^x)$. Let $M > 0$ be arbitrary. Take $n \in \mathbb{N}$ such that $2n\pi > M$. Then we have $|f(\ln(2n\pi))| = 2n\pi > M$. Therefore $f'(x)$ is unbounded on $\mathbb{R}$.