MATH 117 FALL 2014 LECTURE 33 (Nov. 3, 2014)

Read: Bowman §4.A.

- Differentiability.

DEFINITION 1. *A function f is said to be differentiable at $a \in \mathbb{R}$ if and only if the limit $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists and is finite. In this case the limit is called the derivative of f at a, and denoted $f'(a)$.*

Exercise 1. Prove that a function f is differentiable at $a \in \mathbb{R}$ if and only if the limit $\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists and is finite, and in this case the limit is $f'(a)$.

- Basic differentiable functions.

Example 2. $f(x) \equiv c$ is differentiable at every $a \in \mathbb{R}$ with $f'(a) = 0$; $f(x) = x$ is differentiable at every $a \in \mathbb{R}$ with $f'(a) = 1$.

Proof. For $f(x) \equiv c$, we have $\frac{f(x) - f(a)}{x - a} = \frac{c - c}{x - a} = 0$ for every $x \neq a$. Consequently

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} 0 = 0. \tag{1}$$

For $f(x) = x$, we have $\frac{f(x) - f(a)}{x - a} = \frac{x - a}{x - a} = 1$ for every $x \neq a$. Consequently

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} 1 = 1. \tag{2}$$

Thus ends the proofs. □

- Combinations of functions.

THEOREM 3. *Let f, g be differentiable at $a \in \mathbb{R}$. Then*

a) $f \pm g$ is differentiable at a with $(f \pm g)'(a) = f'(a) \pm g'(a)$;

b) $f g$ is differentiable at a with $(f g)'(a) = f'(a)\, g(a) + f(a)\, g'(a)$;

c) If $g(a) \neq 0$, then $\frac{f}{g}$ is differentiable at a with $\left(\frac{f}{g}\right)'(a) = \frac{f'(a)\, g(a) - f(a)\, g'(a)}{g(a)^2}$.

Proof. .

a) As $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$ and $\lim_{x \to a} \frac{g(x) - g(a)}{x - a} = g'(a)$ we have

$$\begin{aligned}
\lim_{x \to a} \frac{(f \pm g)(x) - (f \pm g)(a)}{x - a} &= \lim_{x \to a} \frac{[f(x) \pm g(x)] - [f(a) \pm g(a)]}{x - a} \\
&= \lim_{x \to a} \left[\frac{f(x) - f(a)}{x - a} \pm \frac{g(x) - g(a)}{x - a} \right] \\
&= \left[\lim_{x \to a} \frac{f(x) - f(a)}{x - a} \right] \pm \left[\lim_{x \to a} \frac{g(x) - g(a)}{x - a} \right] \\
&= f'(a) \pm g'(a).
\end{aligned}$$

b) We have

$$\begin{aligned}\lim_{x\to a}\frac{f(x)\,g(x)-f(a)\,g(a)}{x-a} &= \lim_{x\to a}\frac{[f(x)-f(a)]\,g(x)+f(a)\,[g(x)-g(a)]}{x-a}\\ &= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot g(x)\right]+\lim_{x\to a}\left[f(a)\cdot\frac{g(x)-g(a)}{x-a}\right]\\ &= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot g(x)\right]+\left[\lim_{x\to a}f(a)\right]\cdot\left[\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\right]\\ &= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot g(x)\right]+f(a)\,g'(a). \qquad (3)\end{aligned}$$

To proceed we need the following lemma:

LEMMA 4. *Let f be differentiable at $a\in\mathbb{R}$. Then f is continuous at a.*

Proof. (OF THE LEMMA) We have

$$\lim_{x\to a}[f(x)-f(a)]=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot(x-a)=f'(a)\cdot 0=0. \qquad (4)$$

Therefore $\lim_{x\to a}f(x)=\lim_{x\to a}f(a)=f(a)$ and continuity follows. □

With help of the above lemma we have

$$\lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\cdot g(x)\right]=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}g(x)=f'(a)\,g(a) \qquad (5)$$

and the conclusion follows.

c) We have

$$\begin{aligned}\lim_{x\to a}\frac{\left(\frac{f}{g}\right)(x)-\left(\frac{f}{g}\right)(a)}{x-a} &= \lim_{x\to a}\left[\frac{f(x)\,g(a)-f(a)\,g(x)}{x-a}\cdot\frac{1}{g(x)\,g(a)}\right]\\ &= \lim_{x\to a}\left\{\left[\frac{f(x)-f(a)}{x-a}\cdot g(a)-f(a)\cdot\frac{g(x)-g(a)}{x-a}\right]\cdot\frac{1}{g(x)\,g(a)}\right\}\\ &= \left[\lim_{x\to a}\frac{f(x)-f(a)}{x-a}\cdot\lim_{x\to a}g(a)-\lim_{x\to a}f(a)\cdot\lim_{x\to a}\frac{g(x)-g(a)}{x-a}\right]\cdot\\ &\quad\cdot\lim_{x\to a}\frac{1}{g(x)\,g(a)}\\ &= \frac{f'(a)\,g(a)-f(a)\,g'(a)}{g(a)^2}. \qquad (6)\end{aligned}$$

Thus ends the proofs. □

Exercise 2. Point out where was the assumption $g(a)\neq 0$ used in the above proof.

- Polynomials and Rational Functions.
  - From the above we see that polynomials are differentiable everywhere and rational functions $\frac{P(x)}{Q(x)}$ are differentiale wherever $Q\neq 0$.
  - Calculation.

The key formula for calculation of derivatives for rational functions is

$$(x^n)' = n\, x^{n-1} \tag{7}$$

which holds true for all $n \in \mathbb{Z}$.

Example 5. Prove that $(x^3)' = 3\, x^2$.

Proof. Let $f(x) = x^3$. We need to prove $f'(a) = 3\, a^2$ for every $a \in \mathbb{R}$.

- Method 1.

  We know $x' = 1$. Therefore $(x^2)' = x' \cdot x + x \cdot x' = 2\, x$ and $(x^3)' = (x^2)' \cdot x + x^2 \cdot x' = 3\, x^2$.

- Method 2.

  We have

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{(a+h)^3 - a^3}{h} \\ &= \lim_{h \to 0} \frac{[a^3 + 3\, a^2\, h + 3\, a\, h^2 + h^3] - a^3}{h} \\ &= \lim_{h \to 0} \frac{3\, a^2\, h + 3\, a\, h^2 + h^3}{h} \\ &= \lim_{h \to 0} [3\, a^2 + 3\, a\, h + h^2] \\ &= 3\, a^2. \end{aligned}$$

□

Exercise 3. Prove (7).