MATH 117 FALL 2014 LECTURE 31 (OCT. 30, 2014)

- Trigonometric functions.

PROPOSITION 1. *$\sin x$ and $\cos x$ are continuous at every $a \in \mathbb{R}$ if and only if $\lim_{x \to 0+} \sin x = 0$ and $\lim_{x \to 0+} \cos x = 1$.*

Exercise 1. Prove this. (Hint:[1])

Remark 2. The proof of $\lim_{x \to 0+} \sin x = 0$ and $\lim_{x \to 0+} \cos x = 1$ is usually done through geometric argument, as Newton did. However this is against our philosophy here so we will skip it and just accept these as true.

Exercise 2. Prove that $\lim_{x \to a} \frac{\sin a}{\cos a}$ does not exist when $\cos a \neq 0$.

- Intermediate Value Theorem.

THEOREM 3. *Let $f(x)$: $[a, b] \mapsto \mathbb{R}$ be continuous. Then for every s between $f(a)$ and $f(b)$ there is at least one $c \in [a, b]$ such that $f(c) = s$.*

Proof. First the cases $s = f(a)$ or $s = f(b)$ are trivial. Thus in the following we consider $f(a) < s < f(b)$ and $f(b) < s < f(a)$ can be proved almost identically.

Define

$$A := \{x \in [a, b] | \forall y \geqslant x, f(y) > s\}. \tag{1}$$

Clearly $b \in A$ so A is not empty. Set $c := \inf A$. We try to prove $f(c) = s$.

- We prove $f(c) \leqslant s$. Assume the contrary, that is $f(c) > s$. As f is continuous there is $\delta > 0$ such that $\forall |x - c| < \delta$, $|f(x) - f(c)| < f(c) - s$. Note that this means for such x we have $f(x) > s$. Let $c - \delta < x_0 < c$. Then for all $x_0 \leqslant y \leqslant c$, we have $f(y) > s$. On the other hand, for every $y > c$, there is $y_0 \in A$ such that $y_0 < y$ and therefore $f(y) > s$ by the definition of A. Summarizing, we see that $y \geqslant x_0 \Longrightarrow f(y) > s$ and consequently $x_0 \in A$. But this contradicts $c = \inf A$.
- We prove $f(c) \geqslant s$. As $c := \inf A$, there is $x_n \in A$ such that $\lim_{n \to \infty} x_n = c$. By definition of A we have $f(x_n) > s$. Thus by Comparison Theorem and the continuity of f we have

$$f(c) = \lim_{n \to \infty} f(x_n) \geqslant \lim_{n \to \infty} s = s. \tag{2}$$

Thus $f(c) \leqslant s$ and $f(c) \geqslant s$ are both true and consequently $f(c) = s$. □

- Applications of IVT.

Example 4. Let $f(x) := x^5 - 7x^4 + 2x^3 + 3x + 2$. Prove that f has at least one real root. That is there is $c \in \mathbb{R}$ such that $f(c) = 0$.

Proof. We have $f(0) = 2 > 0$ and $f(-1) = -11 < 0$. Furthermore $f(x)$ is a polynomial and is therefore continuous on $[-1, 0]$. Application of IVT now gives the existence of $s \in [-1, 0]$ such that $f(s) = 0$. □

Exercise 3. Prove that the equation $7x^6 - 9x^5 - 1 = 0$ has lt least two real solutions.

---

1. Use the formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$.