MATH 117 FALL 2014 LECTURE 30 (OCT. 29, 2014)

- Continuity of composite function.

THEOREM 1. *Let f be continuous at $a \in \mathbb{R}$ and g be continuous at $f(a) \in \mathbb{R}$. Then $g \circ f$ is continuous at a.*

Proof. Let $\varepsilon > 0$ be arbitrary.

As g is continuous at $f(a)$, there is $\delta_1 > 0$ such that $\forall |y - f(a)| < \delta_1$, $|g(y) - g(f(a))| < \varepsilon$. As f is continuous at a, there is $\delta > 0$ such that $\forall |x - a| < \delta$, $|f(x) - f(a)| < \delta_1$.

Now for every $|x - a| < \delta$, we have $|f(x) - f(a)| < \delta_1$ which leads to $|g(f(x)) - g(f(a))| < \varepsilon$. The proof thus ends. □

- Exponential function.

We define the exponential function for $x \in \mathbb{R}$ through

$$E(x) := \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{nx}. \tag{1}$$

Remark 2. Note that the issue is quite subtle here as we have to ask: How is $\left(1 + \frac{1}{n}\right)^{nx}$ defined? In particular, the usual definition $a^x := e^{x \ln a}$ is not appropriate here.

Problem 1. Try to define $\left(1 + \frac{1}{n}\right)^{nx}$ appropriately so that we do not fall into circular reasoning.

Problem 2. Prove that $E(x)$ exists for every $x \in \mathbb{R}$.

LEMMA 3. *$E(0) = 1$; $\quad E(x + y) = E(x)\, E(y)$.*

Proof. We have

$$E(0) = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n \cdot 0} = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^0 = \lim_{n \to \infty} 1 = 1. \tag{2}$$

We have

$$E(x + y) = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n(x+y)} = \lim_{n \to \infty} \left[\left(1 + \frac{1}{n}\right)^{nx} \cdot \left(1 + \frac{1}{n}\right)^{ny}\right] = \left[\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{nx}\right] \cdot \left[\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{ny}\right] = E(x)\, E(y). \tag{3}$$

Note that the above argument is legitimate only because $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{nx}$ and $\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{ny}$ both exists. □

PROPOSITION 4. *If $\lim_{x \to 0+} E(x) = 1$, then $E(x)$ is continuous at every $a \in \mathbb{R}$.*

Proof. First we show that if $\lim_{x \to 0} E(x) = 1$ then $E(x)$ is continuous at every $a \in \mathbb{R}$. Let $a \in \mathbb{R}$ be arbitrary. Then we have

$$\lim_{x \to a} E(x) = \lim_{x \to a} [E(x - a)\, E(a)] = E(a) \lim_{x \to a} E(x - a) = E(a) \lim_{t \to 0} E(t) = E(a). \tag{4}$$

Next we prove that $\lim_{x \to 0+} E(x) = 1$, then $\lim_{x \to 0-} E(x) = 1$. Once this is done $\lim_{x \to 0} E(x) = 1$ immediately follows (see exercise below). As $E(x)\, E(-x) = 1$, we have

$$\lim_{x \to 0-} E(x) = \lim_{x \to 0-} \frac{1}{E(-x)} = \frac{1}{\lim_{t \to 0+} E(t)} = \frac{1}{1} = 1. \tag{5}$$

Thus ends the proof. $\square$

Exercise 1. Prove by definition that $\lim_{x\to a}E(x-a)=\lim_{t\to 0}E(t)$.

Exercise 2. Prove by definition that $\lim_{x\to a}f(x)=L$ if and only if $\lim_{x\to a+}f(x)=\lim_{x\to a-}f(x)=L$.

PROPOSITION 5. $\lim_{x\to 0+}E(x)=1$.

Remark 6. Note that the following argument is not correct:

$$\lim_{x\to 0+}E(x)=\lim_{x\to 0+}\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nx}\right]=\lim_{n\to\infty}\left[\lim_{x\to 0+}\left(1+\frac{1}{n}\right)^{nx}\right]=\lim_{n\to\infty}1=1. \tag{6}$$

Problem 3. Study the two double limits:

$$\lim_{n\to\infty}\left[\lim_{m\to\infty}[\cos(2\,\pi\,m!\,x)]^n\right];\qquad \lim_{m\to\infty}\left[\lim_{n\to\infty}[\cos(2\,\pi\,m!\,x)]^n\right]. \tag{7}$$

Which one equals the Dirichlet function $D(x)=\begin{cases}1 & x\in\mathbb{Q}\\ 0 & x\notin\mathbb{Q}\end{cases}$.

Proof. (OF THE PROPOSITION) Recall that we have proved before

$$\forall n\in\mathbb{N},\qquad \left(1+\frac{1}{n}\right)^n<4. \tag{8}$$

Thus we have

$$1<\left(1+\frac{1}{n}\right)^{nx}<4^x. \tag{9}$$

Application of Comparison Theorem gives

$$1\leqslant E(x)\leqslant 4^x. \tag{10}$$

Now apply Squeeze Theorem we have

$$\lim_{x\to 0+}E(x)=1. \tag{11}$$

$\square$

Problem 4. Prove $\lim_{x\to 0+}4^x=1$. Note that you have to appropriately define the function 4^x first.