Math 117 Fall 2014 Midterm Exam 2

Oct. 24, 2014 10am - 10:50am. Total 20+2 Pts

NAME: ID#:

- There are five questions.
- Please write clearly and show enough work.

1	
2	
3	
4	
5	
Total	

Question 1. (5 pts) *Prove* ***by definition***:

$$\lim_{x\to 1} x^4 = 1. \tag{1}$$

Proof. Let $\varepsilon > 0$ be arbitrary. Take $\delta = \min\left\{1, \frac{\varepsilon}{15}\right\}$. For every $0 < |x-1| < \delta$, we have $|x-1|<1$ and therefore $|x| \leqslant 1 + |x-1| < 2$. Now for such x we have

$$|x^4 - 1| = |x-1|\,|x^2+1|\,|x+1| < \delta\,[|x|^2+1]\,[|x|+1] < 15\,\delta \leqslant \varepsilon. \tag{2}$$

Thus ends the proof. □

Question 2. (5 pts) *Prove* ***by definition****:*

$$\lim_{n\to\infty} \frac{3^n}{n} = +\infty. \tag{3}$$

Proof. (Method 1) First prove by induction $\forall n \in \mathbb{N}, n \leqslant 2^n$.

- $n = 1$: We have $1 \leqslant 2^1 = 2$;
- Assume $k \leqslant 2^k$. Then as $k \geqslant 1$, we have $\frac{k+1}{k} \leqslant 2$ and $k + 1 = \frac{k+1}{k} \cdot k \leqslant 2\,k \leqslant 2 \cdot 2^k = 2^{k+1}$.

Now let $M > 0$ be arbitrary. Take $N \in \mathbb{N}$ such that $N > \log_{3/2} M$. Then for every $n \geqslant N$ we have

$$\frac{3^n}{n} \geqslant \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n \geqslant \left(\frac{3}{2}\right)^N > \left(\frac{3}{2}\right)^{\log_{3/2} M} = M. \tag{4}$$

Thus ends the proof. □

Proof. (Method 2) First we apply binomial expansion to obtain:

$$3^n = (1+2)^n = \binom{n}{0} 1^n \cdot 2^0 + \binom{n}{1} 1^{n-1} \cdot 2^1 + \binom{n}{2} 1^{n-2} \cdot 2^2 + \cdots + \binom{n}{n} 1^0 \cdot 2^n > \binom{n}{2} 1^{n-2} \cdot 2^2 = 2\,n\,(n-1). \tag{5}$$

Now let $M > 0$ be arbitrary. Take $N \in \mathbb{N}$ such that $N > \frac{M}{2} + 1$. Then for every $n \geqslant N$, we have

$$\frac{3^n}{n} > \frac{2\,n\,(n-1)}{n} = 2\,(n-1) \geqslant 2\,(N-1) > M. \tag{6}$$

Thus ends the proof. □

Question 3. (5 pts) *Let* $a_n = (-1)^n - \frac{\sin n^2}{n}$. *Calculate* $\liminf_{n \to \infty} a_n$ *and justify your answer.*

Solution. Let

$$m_n := \inf_{k \geqslant n} a_n = \inf \left\{ (-1)^n - \frac{\sin n^2}{n}, (-1)^{n+1} - \frac{\sin(n+1)^2}{n+1}, \ldots \right\}. \tag{7}$$

We prove

- $m_n \geqslant -1 - \frac{1}{n}$. Let $k \geqslant n$ be arbitrary. We have $(-1)^k \geqslant -1$, $-\frac{\sin k^2}{k} \geqslant -\frac{1}{n}$. Therefore $a_k \geqslant -1 - \frac{1}{n}$. So $-1 - \frac{1}{n}$ is a lower bound for $\left\{ (-1)^n - \frac{\sin n^2}{n}, (-1)^{n+1} - \frac{\sin(n+1)^2}{n+1}, \ldots \right\}$ and consequently $m_n \geqslant -1 - \frac{1}{n}$.

- $m_n \leqslant -1 + \frac{1}{n}$. We have

$$m_n \leqslant a_{2n+1} = (-1)^{2n+1} - \frac{\sin(2\,n+1)^2}{(2\,n+1)} \leqslant -1 + \frac{1}{2\,n+1} < -1 + \frac{1}{n}. \tag{8}$$

- $\lim_{n \to \infty} \left(-1 + \frac{1}{n} \right) = -1$. Let $\varepsilon > 0$ be arbitrary. Take $N \in \mathbb{N}$ such that $N > \varepsilon^{-1}$. Then for every $n \geqslant N$, we have $\left| \left(-1 + \frac{1}{n} \right) - (-1) \right| = \frac{1}{n} \leqslant \frac{1}{N} < \varepsilon$.

- Similarly $\lim_{n \to \infty} \left(-1 - \frac{1}{n} \right) = -1$.

Thus we have $-1 - \frac{1}{n} \leqslant m_n \leqslant -1 + \frac{1}{n}$ and $\lim_{n \to \infty} \left(-1 + \frac{1}{n} \right) = -1$, $\mathrm{im}_{n \to \infty} \left(-1 - \frac{1}{n} \right) = -1$. It now follows from Squeeze that $\lim_{n \to \infty} m_n = -1$ and now by definition $\liminf_{n \to \infty} a_n = -1$.

Question 4. (5 pts) *Let* $\{a_n\}$ *be increasing and not Cauchy. Prove that* $\lim_{n\to\infty} a_n = +\infty$.

Proof. We claim that $\{a_n\}$ is not bounded above. Since otherwise $\{a_n\}$ converges and then is Cauchy.

Now we prove $\lim_{n\to\infty} a_n = +\infty$. Let $M > 0$ be arbitrary. As $\{a_n\}$ is not bounded above, there is $n_0 \in \mathbb{N}$ such that $a_{n_0} > M$. Now set $N = n_0$. Then for every $n \geqslant N$, we have $a_n \geqslant a_{n_0} > M$. □

Question 5. (Extra 2 pts) *Let* $f(x), g(x): \mathbb{R} \mapsto \mathbb{R}$ *and* $a, b, L \in \mathbb{R}$. *Assume* $\lim_{x \to a} f(x) = b$ *and* $\lim_{x \to b} g(x) = L$. *Prove or disprove:* $\lim_{x \to a} g(f(x)) = L$.

Solution. The claim is not true. Consider $f(x) = 0$ for all x and $g(x) = \left\{ \begin{array}{ll} 0 & x \neq 0 \\ 1 & x = 0 \end{array} \right\}$. Now we have

$$\lim_{x \to 0} f(x) = 0, \qquad \lim_{x \to 0} g(x) = 0 \tag{9}$$

but $g(f(x)) = g(0) = 1$ which means

$$\lim_{x \to 0} g(f(x)) = 1 \neq 0. \tag{10}$$

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