# Math 117 Fall 2014 Homework 6 Solutions

## Due Thursday Oct. 30 3pm in Assignment Box

Question 1. (5 pts) *Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series. Prove that*

$$\sum_{n=1}^{\infty} a_n \text{ converges} \Longrightarrow \lim_{n\to\infty} a_n = 0. \tag{1}$$

**Proof.** Let $\varepsilon > 0$ be arbitrary. As $\sum_{n=1}^{\infty} a_n$ converges, it is Cauchy. Thus there is $N_1 \in \mathbb{N}$ such that for every $m > n \geqslant N_1$,

$$\left|\sum_{k=n+1}^{m} a_k\right| < \varepsilon. \tag{2}$$

Now set $N = N_1 + 1$. For every $n \geqslant N$, we have $n > n - 1 \geqslant N$ and consequently

$$|a_n| = \left|\sum_{k=(n-1)+1}^{n} a_k\right| < \varepsilon. \tag{3}$$

Thus by definition $\lim_{n\to\infty} a_n = 0$. □

Question 2. (5 pts) *Let* $r, c \in \mathbb{R}$. *Prove that* $\sum_{n=1}^{\infty} c\, r^n$ *converges if and only if* $|r| < 1$. *(You can use the conclusion of Question 1).*

**Proof.** It suffices to prove that $|r| < 1 \Longrightarrow$ convergence and $|r| \geqslant 1 \Longrightarrow$ divergence.

- $|r| < 1$. We prove $\sum_{n=1}^{\infty} c\, r^n$ is Cauchy. Let $\varepsilon > 0$ be arbitrary. Take $N \in \mathbb{N}$ such that $|c|\,|r|^N < \varepsilon\,(1 - r)$. Now for every $m > n \geqslant N$, we have

$$\left|\sum_{k=n+1}^{m} a_k\right| = |c\, r^{n+1}|\,|1 + r + \cdots + r^{m-n-1}| = |c\, r^{n+1}|\,\frac{|1 - r^{m-n}|}{|1 - r|} < \frac{|c|\,|r|^N}{1 - r} < \varepsilon. \tag{4}$$

  Therefore the series converges.

- $|r| \geqslant 1$. We claim that $\lim_{n\to\infty} c\, r^n = 0$ is false in this case. Thus the series diverges following Question 1.

  Let $N \in \mathbb{N}$ be arbitrary. Take $n = N \geqslant N$. Then

$$|c\, r^n| = |c|\,|r|^N \geqslant |c|. \tag{5}$$

  Thus $\lim_{n\to\infty} c\, r^n = 0$ is false by definition. □

**Remark.** (To Grader) There is in fact a situation where $|r| \geqslant 1$ but the series is convergent, namely $c = 0$. Anyone noticing this should get one extra point.

Question 3. (5 pts) *Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series. Prove: If there is* $b_n \geqslant 0$ *such that* $\sum_{n=1}^{\infty} b_n$ *converges and* $\forall n \in \mathbb{N}\; |a_n| \leqslant b_n$, *then* $\sum_{n=1}^{\infty} a_n$ *converges.*

**Proof.** We prove that $\sum_{n=1}^{\infty} a_n$ is Cauchy.

Let $\varepsilon > 0$ be arbitrary. As $\sum_{n=1}^{\infty} b_n$ is convergent, there is $N_1 \in \mathbb{N}$ such that $\forall m > n \geqslant N_1$, $\left|\sum_{k=n+1}^{m} b_k\right| < \varepsilon$. Take $N = N_1$. Then for every $m > n \geqslant N$, we have

$$\left|\sum_{k=n+1}^{m} a_k\right| \leqslant \sum_{k=n+1}^{m} |a_k| \leqslant \sum_{k=n+1}^{m} b_k = \left|\sum_{k=n+1}^{m} b_k\right| < \varepsilon. \tag{6}$$

Here we have used triangle inequality and the fact that $b_k \geqslant 0$ for all $k$. □

QUESTION 4. (5 PTS) *Let* $\sum_{n=1}^{\infty} a_n$ *be an infinite series.*

a) (2 PTS) *If* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, *then* $\sum_{n=1}^{\infty} a_n$ *converges;*

b) (2 PTS) *If* $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$, *then* $\sum_{n=1}^{\infty} a_n$ *diverges;*

c) (1 PT) *Find an infinite series satisfying* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$ *and also* $\liminf_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$. *You don't need to justify your claims.*

*(You can use the conclusions from Questions 1 – 3)*

**Solution.**

a) We prove that if $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$, then there is $|r| < 1$ and $c > 0$ such that $\forall n \in \mathbb{N}$, $|a_n| \leqslant c\, r^n$. Once this is done the convergence follows from Question 3.

We denote $r_0 := \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$. As $r_0 < 1$, there is $0 < r < 1$ such that $r > r_0$. Now by definition, there is $N \in \mathbb{N}$ such that $\forall n \geqslant N$,

$$\left|\sup_{k\geqslant n}\left\{\frac{|a_{k+1}|}{|a_k|}\right\} - r_0\right| < r - r_0. \tag{7}$$

Application of triangle inequality gives

$$\sup_{k\geqslant n}\left\{\frac{|a_{k+1}|}{|a_k|}\right\} < r \tag{8}$$

which by definition of sup gives

$$\forall n \geqslant N\ \forall k \geqslant n, \qquad \frac{|a_{k+1}|}{|a_k|} < r. \tag{9}$$

This is equivalent to

$$\forall n \geqslant N, \qquad \frac{|a_{n+1}|}{|a_n|} < r \Longrightarrow |a_{n+1}| < r\, |a_n|. \tag{10}$$

Therefore we have

$$\forall k \in \mathbb{N} \qquad |a_{N+k}| < r\, |a_{N+k-1}| < r^2\, |a_{N+k-2}| < \cdots < r^k\, |a_N|. \tag{11}$$

Now set

$$c := \max\left\{\frac{|a_1|}{r}, \frac{|a_2|}{r^2}, \ldots, \frac{|a_N|}{r^N}\right\}. \tag{12}$$

Then by definition of $c$ we have

$$\forall n \leqslant N, \qquad |a_n| \leqslant c\, r^n. \tag{13}$$

On the other hand, for every $n > N$, we have

$$|a_n| < r^{n-N}\, |a_N| \leqslant r^{n-N} c\, r^N = c\, r^n. \tag{14}$$

Therefore $\forall n \in \mathbb{N}$, $|a_n| < c\,r^n$ and convergence follows.

b) Denote $r_0 := \liminf_{n\to\infty} \frac{|a_{n+1}|}{|a_n|}$. By definition there is $N_1 \in \mathbb{N}$ such that for all $n \geqslant N_1$,

$$\left| \inf_{k \geqslant n} \left\{ \frac{|a_{k+1}|}{|a_k|} \right\} - r_0 \right| < r_0 - 1. \tag{15}$$

Application of triangle inequality gives

$$\forall n \geqslant N_1, \forall k \geqslant n, \qquad \frac{|a_{k+1}|}{|a_k|} > 1 \tag{16}$$

or equivalently

$$\forall n \geqslant N_1, \qquad \frac{|a_{n+1}|}{|a_n|} > 1. \tag{17}$$

This gives

$$|a_{N_1}| < |a_{N_1+1}| < |a_{N_1+2}| < \cdots. \tag{18}$$

Note that as $\frac{|a_{N_1+1}|}{|a_{N_1}|}$ is well-defined, $a_{N_1} \neq 0 \Longrightarrow |a_{N_1}| > 0$.

We prove that $\lim_{n\to\infty} a_n = 0$ does not hold and divergence then follows. Let $N \in \mathbb{N}$ be arbitrary. Take $n := \max\{N_1, N\} \geqslant N$. Then we have

$$|a_n| \geqslant |a_{N_1}| > 0. \tag{19}$$

So by definition $\lim_{n\to\infty} a_n = 0$ does not hold.

c) For example $\sum_{n=1}^{\infty} a_n$ with $a_n = \begin{cases} \frac{1}{n} & n \text{ odd} \\ \frac{2}{n} & n \text{ even} \end{cases}$.