MATH 117 FALL 2014 LECTURE 24 (OCT. 16, 2014)

Reading:

- Let $\{a_n\}$ be a bounded sequence. Recall:
  - Can define the set of accumulation points:

$$A(\{a_n\}) := \{a \in \mathbb{R} \mid \exists a_{n_k} \longrightarrow a\}. \tag{1}$$

  - Have seen:

$$A(\{(-1)^n\}) = \{1, -1\}; \qquad A\big(\{\{n\sqrt{2}\}\}\big) = [0, 1]. \tag{2}$$

Exercise 1. Let $a_n = (-1)^n + \{n\sqrt{2}\}$. Find $A(\{a_n\})$ and justify.

Exercise 2. Construct a sequence $\{a_n\}$ such that $A(\{a_n\}) = [0, 1] \cup [2, 4]$.

  - Question: What are $\sup A$ and $\inf A$? Can we represent them using a_n?

- Limit superior and Limit inferior.
  - Let $\{a_n\}$ be a bounded sequence. Define

$$M_n := \sup\{a_n, a_{n+1}, \ldots\}, \qquad m_n := \inf\{a_n, a_{n+1}, \ldots\}. \tag{3}$$

Exercise 3. Prove that $\{M_n\}$ is decreasing and $\{m_n\}$ is increasing.

Define the limit superior of $\{a_n\}$ as

$$\limsup_{n\to\infty} a_n := \lim_{n\to\infty} M_n \tag{4}$$

and the limit inferior of $\{a_n\}$ as

$$\liminf_{n\to\infty} a_n := \lim_{n\to\infty} m_n. \tag{5}$$

Exercise 4. Why do the two limits exist?

Exercise 5. Define $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ for a sequence $\{a_n\}$ not necessarily bounded.

  - Today's main theorem.

THEOREM 1. *We have*

$$M = \sup A(\{a_n\}), \qquad m = \inf A(\{a_n\}). \tag{6}$$

Proof. We prove the first claim and leave the second one as exercise. In the following we simply write A instead of $A(\{a_n\})$.

- M is an upper bound for A.
  Let $a \in A$ be arbitrary. Then by definition there is a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = a$. Now clearly

$$a_{n_k} \leqslant \sup\{a_{n_k}, a_{n_k+1}, a_{n_k+2}, \ldots\} = M_{n_k}, \tag{7}$$

  from which it follows $\lim_{k\to\infty} a_{n_k} \leqslant \lim_{k\to\infty} M_{n_k}$, that is $a \leqslant M$.

- M is the least upper bound of A.
  Let $m < M$ be arbitrary. Then $\frac{m+M}{2} < M \leqslant M_n$ for every $n \in \mathbb{N}$.
  - As $\frac{m+M}{2} < M_1$, it is not an upper bound of $\{a_1, a_2, \ldots\}$. Thus there is $a_{n_1} \geqslant \frac{m+M}{2}$;

- As $\frac{m+M}{2} < M_{n_1+1}$, it is not an upper bound of $\{a_{n_1+1},\ a_{n_1+2},\ \ldots\}$. Therefore there is $n_2 > n_1$ such that $a_{n_2} > \frac{m+M}{2}$;
- Repeating this process we have a subsequence $\{a_{n_k}\}$ such that

$$\forall k \in \mathbb{N}, \qquad a_{n_k} \geqslant \frac{m+M}{2}. \tag{8}$$

Now since $\{a_n\}$ is bounded, so is $\{a_{n_k}\}$. Application of Bolzano-Weierstrass gives the existence of a convergent subsequence $\left\{a_{n_{k_l}}\right\}$. Set $a := \lim_{l \to \infty} a_{n_{k_l}}$. Application of comparison now gives

$$a \geqslant \frac{m+M}{2} > m \tag{9}$$

which means m is not an upper bound of A.

Thus ends the proof. □

COROLLARY 2. *Let $\{a_n\}$ be a bounded sequence. Then it converges if and only if* $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n$.

Remark 3. Let $\{a_n\}$ be a bounded sequence. Then its limit superior and limit inferior always exist (in contrast to its limit).

Exercise 6. What happens if $\{a_n\}$ is not bounded? Does the conclusion of the corollary still hold (with appropriate interpretation)?

○ Calculating limsup and liminf.

Example 4. Let $a_n = (-1)^n + \frac{1}{n^2}$. Calculate $\limsup_{n\to\infty} a_n$ and justify your answers.
Solution. We have by definition

$$M_n = \sup\left\{(-1)^n + \frac{1}{n^2}, (-1)^{n+1} + \frac{1}{(n+1)^2}, \ldots\right\}. \tag{10}$$

As $(-1)^n + \frac{1}{n^2} \leqslant 1 + \frac{1}{n^2}$ for every n, $1 + \frac{1}{n^2}$ is an upper bound of the set and we have

$$M_n \leqslant 1 + \frac{1}{n^2}. \tag{11}$$

On the other hand, $1 \leqslant (-1)^{2n} + \frac{1}{(2n)^2} \in \left\{(-1)^n + \frac{1}{n^2}, (-1)^{n+1} + \frac{1}{(n+1)^2}, \ldots\right\}$ which gives

$$1 \leqslant M_n. \tag{12}$$

Therefore $1 \leqslant M_n \leqslant 1 + \frac{1}{n^2}$ and we have $\lim_{n\to\infty} M_n = 1$ by Squeeze Theorem.

Remark 5. For this particular example we can actually find M_n exactly, as shown below. Note that as all we need is $\lim_{n\to\infty} M_n$, calculating M_n exactly is usually not the most efficient way to go.

Recall that $M_n := \sup\{a_n, a_{n+1}, \ldots\} = \sup_{k\geqslant n} a_k = \sup_{k\geqslant n}\left\{(-1)^k + \frac{1}{k^2}\right\}$. We discuss two cases.

– n odd.

In this case we claim that $M_n = (-1)^{n+1} + \frac{1}{(n+1)^2} = 1 + \frac{1}{(n+1)^2}$. As $1 + \frac{1}{(n+1)^2} \in \{a_n, a_{n+1}, \ldots\}$, to show it is the supreme we only need to show it is an upper bound. Let $k \geqslant n$ be arbitrary. If k is odd, then

$$a_k = -1 + \frac{1}{k^2} \leqslant 0 \leqslant 1 + \frac{1}{(n+1)^2}. \tag{13}$$

If k is even, then $k \geqslant (n+1)$ and

$$a_k = 1 + \frac{1}{k^2} \leqslant 1 + \frac{1}{(n+1)^2}. \tag{14}$$

Therefore $\forall k \geqslant n,\ a_k \leqslant 1 + \frac{1}{(n+1)^2}$.

– n even.

In this case we claim that $M_n = (-1)^n + \frac{1}{n^2} = 1 + \frac{1}{n^2}$. Let $k \geqslant n$ be arbitrary. Then we have

$$a_k = (-1)^k + \frac{1}{k^2} \leqslant 1 + \frac{1}{k^2} \leqslant 1 + \frac{1}{n^2}. \tag{15}$$

Therefore $\forall k \geqslant n,\ a_k \leqslant 1 + \frac{1}{n^2}$.

Exercise 7. Calculate $\liminf_{n\to\infty} a_n$ and justify your answers.

Exercise 8. Calculate $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ for $a_n = (-1)^{n^2} - e^{-n}$.