MATH 117 FALL 2014 LECTURE 22 (OCT. 10, 2014)

Reading: Bowman §3.D; 314 Limit & Continuity §3.

- Function limit and sequence limit.

THEOREM 1. *Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ and $a, L \in \mathbb{R}$. Then $\lim_{x \to a} f(x) = L$ if and only if for every sequence $\{x_n\}$ satisfying $\lim_{n \to \infty} x_n = a$ and $\forall n \in \mathbb{N}, x_n \neq a$, $\lim_{n \to \infty} f(x_n) = L$.*

Proof. We first prove "only if" then prove "if".

- "Only if". Assume $\lim_{x \to a} f(x) = L$. Let $\{x_n\}$ be an arbitrary sequence satisfying the conditions. Let $\varepsilon > 0$ be arbitrary.

  As $\lim_{x \to a} f(x) = L$, there is $\delta > 0$ such that for every $0 < |x - a| < \delta$, $|f(x) - L| < \varepsilon$;

  As $\lim_{n \to \infty} x_n = a$, there is $N \in \mathbb{N}$ such that for every $n \geqslant N$, $|x_n - a| < \delta$;

  As $\forall n \in \mathbb{N}, x_n \neq a$, there holds for every n, $0 < |x_n - a|$.

  Summarizing, we see that for every $n \geqslant N$, we have $0 < |x_n - a| < \delta$ which yields $|f(x_n) - L| < \varepsilon$. Thus ends the proof for "Only if".

- "If". Assume that for every sequence $\{x_n\}$ satisfying $\lim_{n \to \infty} x_n = a$ and $\forall n \in \mathbb{N}, x_n \neq a$, $\lim_{n \to \infty} f(x_n) = L$. We prove $\lim_{x \to a} f(x) = L$ through proof by contradiction.

  Assume the contrary. Then there is $\varepsilon > 0$ such that $\forall \delta > 0$, there is $0 < |x - a| < \delta$ satisfying $|f(x) - L| \geqslant \varepsilon$.

  Take $\delta_1 = 1$. Then there is x_1 satisfying $0 < |x_1 - a| < 1$ with $|f(x_1) - L| \geqslant \varepsilon$.

  Take $\delta_2 = \min \left\{ \frac{1}{2}, |x_1 - a| \right\}$. Then there is x_2 satisfying $0 < |x_2 - a| < \delta_2 \leqslant \frac{1}{2}$ with $|f(x_2) - L| \geqslant \varepsilon$.

  Repeating this we obtain a sequence $\{x_n\}$ satisfying $0 < |x_n - a| < \frac{1}{n}$ and $|f(x_n) - L| \geqslant \varepsilon$ for every $n \in \mathbb{N}$.

  As $0 < |x_n - a|$ we see that $\forall n \in \mathbb{N}$, $x_n \neq a$;

  As $|x_n - a| < \frac{1}{n}$ we have $-\frac{1}{n} < x_n - a < \frac{1}{n}$ which yields $\lim_{n \to \infty} x_n = a$ thanks to Squeeze Theorem;

  Now we prove that $\lim_{n \to \infty} f(x_n) = L$ does not hold. We still use the above particular ε. Let $N \in \mathbb{N}$ be arbitrary. Take $n = N + 1$. Then we have $n \geqslant N$ and $|f(x_n) - L| \geqslant \varepsilon$. ☐

Theorem 1 has the following variant where the limit L does not appear explicitly.

THEOREM 2. *Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ and $a \in \mathbb{R}$. Then $\lim_{x \to a} f(x)$ exists if and only if for every sequence $\{x_n\}$ satisfying $\lim_{n \to \infty} x_n = a$ and $\forall n \in \mathbb{N}, x_n \neq a$, $\lim_{n \to \infty} f(x_n)$ exists.*

Problem 1. Prove Theorem Theorem 2.

Exercise 1. Does the conclusion of Theorem 2 still holds if we replace "$\lim_{n \to \infty} f(x_n)$ exists" by "$\{f(x_n)\}$ is Cauchy"?

Exercise 2. Generalize the Theorems 1 and 2 to the situation $f(x): A \mapsto \mathbb{R}$ and then prove the generalized version.

Exercise 3. Do the Theorems 1 and 2 still hold if $a = \pm\infty$ or $L = \pm\infty$? Justify your answers.

- The following variant of Theorem 1 is especially convenient in the proof of non-existence of $\lim_{x \to a} f(x)$.

  THEOREM 3. *Let $f(x): \mathbb{R} \mapsto \mathbb{R}$ and $a \in \mathbb{R}$. Then $\lim_{x \to a} f(x)$ does not exist if and only if there are two sequences $\{x_n\}, \{y_n\}$ satisfying $\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = a$ and $\forall n \in \mathbb{N}, x_n \neq a$, $y_n \neq a$, but $\lim_{n \to \infty} f(x_n) = L_1 \neq L_2 = \lim_{n \to \infty} f(y_n)$.*

Exercise 4. Prove Theorem 3.

Example 4. Let $f(x)=\begin{cases} 1 & x>0 \\ 0 & x\leqslant 0 \end{cases}$. Prove that $\lim_{x\to 0}f(x)$ does not exist.

Proof. Take $x_n=\frac{1}{n}$ and $y_n=-\frac{1}{n}$. Then we have $\lim_{n\to\infty}x_n=\lim_{n\to\infty}y_n=0$, $x_n\neq 0$, $y_n\neq 0$ but $\lim_{n\to\infty}f(x_n)=\lim_{n\to\infty}1=1\neq 0=\lim_{n\to\infty}f(y_n)$. Thus ends the proof. □

- Left/right limits.

DEFINITION 5. (LEFT LIMIT) *Let $f:\mathbb{R}\mapsto\mathbb{R}$ and $a\in\mathbb{R}$. We say $f(x)$ has left limit L at a if and only if*

$$\forall\varepsilon>0\ \exists\delta>0\ \forall a-\delta<x<a, \qquad |f(x)-L|<\varepsilon. \tag{1}$$

Exercise 5. Define "Right Limit". (Ans:[1])

NOTATION 6. *We write*

$$\lim_{x\to a-}f(x)=L, \qquad \lim_{x\to a+}f(x)=L \tag{2}$$

for left/right limits respectively.

THEOREM 7. *Let $f:\mathbb{R}\mapsto\mathbb{R}$. $a,L\in\mathbb{R}$. Then $\lim_{x\to a}f(x)=L$ if and only if $\lim_{x\to a-}f(x)=\lim_{x\to a+}f(x)=L$.*

Proof. We prove "If" then "Only if".

- If. Let $\varepsilon>0$ be arbitrary. As $\lim_{x\to a-}f(x)=L$ there is $\delta_L>0$ such that when $a-\delta_L<x<a$ we have $|f(x)-L|<\varepsilon$;

  As $\lim_{x\to a+}f(x)=L$ there is $\delta_R>0$ such that when $a<x<a+\delta_R$, $|f(x)-L|<\varepsilon$.

  Set $\delta=\min\{\delta_L,\delta_R\}$. Then for every $0<|x-a|<\delta$, either $a-\delta_L\leqslant a-\delta<x<a$ or $a<x<a+\delta\leqslant a+\delta_R$. Either case leads to $|f(x)-L|<\varepsilon$.

- Only if. We prove $\lim_{x\to a-}f(x)=L$ and leave the proof of $\lim_{x\to a+}f(x)=L$ as exercise as it is almost identical to the left limit proof.

  Let $\varepsilon>0$ be arbitrary. As $\lim_{x\to a}f(x)=L$ there is $\delta_0>0$ such that $0<|x-a|<\delta_0$ implies $|f(x)-L|<\varepsilon$. Now take $\delta=\delta_0$. For every $a-\delta<x<a$ we have $|x-a|=a-x<\delta=\delta_0$ which means $0<|x-a|<\delta_0$ and consequently $|f(x)-L|<\varepsilon$. Therefore $\lim_{x\to a-}f(x)=L$. □

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1. $\forall\varepsilon>0\ \exists\delta>0\ \forall a<x<a+\delta, \qquad |f(x)-L|<\varepsilon.$