Math 117 Fall 2014 Homework 5 Solutions

Due Thursday Oct. 16 3pm in Assignment Box

Question 1. (5 pts) *Let f, g: $\mathbb{R} \mapsto \mathbb{R}$ and $a \in \mathbb{R}$. Further assume* $\lim_{x \to a} f(x) = L \in \mathbb{R}$ *and* $\lim_{x \to a} g(x) = M \in \mathbb{R}$.

a) (2 pts) *Prove or disprove: Under the above assumptions, there is $M > 0$ such that $\forall x \in \mathbb{R}$, $|f(x)| < M$;*

b) (2 pts) *Prove* ***by definition****:* $\lim_{x \to a} [f(x)\, g(x)] = L\, M$;

c) (1 pt) *Compare your proof with that of $\lim_{n \to \infty} a_n\, b_n = a\, b$ in the lecture note for Oct.6. Is your proof simply a "translation" of the proof there? Are there any new ideas involved? Explain why these new ideas are necessary.*

Proof.

a) The claim is not true. For example let $a = 0$ and $f(x) = x$. We have, on one hand, for every $\varepsilon > 0$, taking $\delta = \varepsilon$ gives $\forall 0 < |x - 0| < \delta$, $|f(x) - 0| = |x - 0| < \delta = \varepsilon$. Therefore $\lim_{x \to 0} f(x) = 0$. On the other hand, let $M > 0$ be arbitrary, taking $x = M$ we have $|f(x)| = M \geqslant M$.

b) Let $\varepsilon > 0$ be arbitrary. As $\lim_{x \to a} f(x) = L$, there is $\delta_1 > 0$ such that for every $0 < |x - a| < \delta_1$, $|f(x) - L| < 1 \Longrightarrow |f(x)| < |L| + 1$ (triangle); Furthermore there is $\delta_2 > 0$ such that for every $0 < |x - a| < \delta_2$, $|f(x) - L| < \frac{\varepsilon}{2\, M}$; As $\lim_{x \to a} g(x) = M$, there is $\delta_3 > 0$ such that for every $0 < |x - a| < \delta_3$, $|g(x) - M| < \frac{\varepsilon}{2\, (|L| + 1)}$. Now take $\delta = \min \{\delta_1, \delta_2, \delta_3\}$, we have for $0 < |x - a| < \delta$,

$$
\begin{aligned}
|f(x) g(x) - L\, M| &= |[f(x) - L]\, M + f(x)\, [g(x) - M]| \\
&< \frac{\varepsilon}{2\, M} + (|L| + 1)\, \frac{\varepsilon}{2\, (|L| + 1)} = \varepsilon.
\end{aligned}
$$

c) Because of a), we have to introduce $\delta_1 > 0$. This is a new idea and is not from translating the sequence proof. □

Question 2. (5 pts) *Study* $\lim_{x \to a} (\sqrt{x + 1} - \sqrt{x})$ *in the following situations:*

a) (2 pts) $a = +\infty$;

b) (3 pts) $a = 0$;

Justify any claim you make.

Solution.

a) Let $\varepsilon > 0$ be arbitrary. Take $M = \varepsilon^{-2}$. Then for every $x > M$, we have

$$|(\sqrt{x + 1} - \sqrt{x}) - 0| = \left| \frac{1}{\sqrt{x + 1} + \sqrt{x}} \right| < \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{M}} = \varepsilon. \tag{1}$$

Therefore the limit is 0.

b) We first prove $\lim_{x \to 0} \sqrt{x} = 0$. Let $\varepsilon > 0$ be arbitrary. Take $\delta = \varepsilon^2$. Then for $0 < x < \delta$ (Note that as the domain of $\sqrt{x + 1} - \sqrt{x}$ is $x \geqslant 0$, $0 < |x - 0| < \delta$ becomes $0 < x < \delta$.) we have

$$|\sqrt{x} - 0| < \sqrt{\delta} = \varepsilon. \tag{2}$$

Next we prove $\lim_{x \to 0} \sqrt{1+x} = 1$. As the domain for the function is $x \geqslant 0$ we only need to consider $x \to 0+$ here. Let $\varepsilon > 0$ be arbitrary. Take $\delta = 2\varepsilon$. We have for $0 < x < \delta$

$$|\sqrt{1+x} - 1| = \sqrt{1+x} - 1 < 1 + \frac{x}{2} - 1 < \frac{\delta}{2} = \varepsilon. \tag{3}$$

Therefore the limit when $x \to 0$ is $1 - 0 = 1$.

To grader: It is OK if discussion on the domain of the function is missing.

QUESTION 3. (5 PTS) *Let $\{a_n\}$ be a bounded sequence. Defined the set A to consist of all the a_n's, that is $A = \{a_1, a_2, a_3, \ldots\}$. Let $M := \sup A$. Prove that*

a) (1 PT) *$M \in \mathbb{R}$.*

b) (4 PTS) *If there is no $n \in \mathbb{N}$ such that $M = a_n$, then there exists a increasing subsequence $\{a_{n_k}\}$ such that $\lim_{k \to \infty} a_{n_k} = M$. Make sure you check the definition for subsequences.*

Proof.

a) Clearly $M \geqslant a_1$ therefore it is either a real number of $+\infty$. As $\{a_n\}$ is bounded, there is $M' > 0$ such that $\forall n \in \mathbb{N}$, $|a_n| < M'$. In particular we have $\forall n \in \mathbb{N}$, $a_n < M'$ that is M' is an upper bound of $\{a_n\}$. Now by definition $M = \sup A \leqslant M'$ that is $M \in \mathbb{R}$.

b) We construct a_{n_k} one by one as follows.

- First take $a_{n_1} = a_1$.
- As $a_1 \neq M$, $m_1 := a_1 < M$. But m_1 cannot be an upper bound for $\{a_n\}$ which means there is $n_2 > 1$ such that $a_{n_2} > m_1 = a_1$.
- Let $m_2 := \max\left\{a_1, \ldots, a_{n_2}, M - \frac{1}{2}\right\}$. We have $m_2 < M$. Therefore m_2 is not an upper bound for $\{a_n\}$ and there must be $n_3 \in \mathbb{N}$ such that $a_{n_3} > m_2$. By definition of m_2 we have $n_3 > n_2$ and $a_{n_3} > a_{n_2}$.
- Let $m_3 := \max\left\{a_1, \ldots, a_{n_3}, M - \frac{1}{3}\right\}$. We can find $n_4 > n_3$ such that $a_{n_4} > a_{n_3}$.

Repeating this process we obtain an increasing subsequence $\{a_{n_k}\}$ satisfying $a_{n_k} > M - \frac{1}{k}$. On the other hand we have $a_{n_k} < M$. Application of Squeeze Theorem gives $\lim_{k \to \infty} a_{n_k} = M$. $\square$

QUESTION 4. (5 PTS) *Let $f: \mathbb{Q} \mapsto \mathbb{R}$ be defined as*

$$f(x) = \frac{1}{q} \qquad \text{when } x = \frac{p}{q}, \ p, q \in \mathbb{Z}, q > 0, (p, q) = 1. \tag{4}$$

Let $a \in \mathbb{R}$. Study $\lim_{x \to a} f(x)$. You need to justify any claim you make.

Solution. We claim $\lim_{x \to a} f(x) = 0$ for every $a \in \mathbb{R}$.

Let $\varepsilon > 0$ be arbitrary. Let

$$A := \left\{x \in \mathbb{Q} | \, x = \frac{p}{q}, p, q \in \mathbb{Z}, q > 0, (p, q) = 1, q \leqslant \varepsilon^{-1}, |x - a| < 1\right\}. \tag{5}$$

Then A is a finite set as there are only finitely many natural numbers $q \leqslant \varepsilon^{-1}$ and for each q there are only finitely many $p \in \mathbb{Z}$ satisfying $\left|\frac{p}{q} - a\right| < 1$. Consequently,

$$\delta_1 := \min_{x \in A, x \neq a} \{|x - a|\} > 0. \tag{6}$$

Set $\delta := \min\{1, \delta_1\}$. Then for any $0 < |x - a| < \delta$, $x \in \mathbb{Q}$, we must have $|x - a| < 1$ but $x \notin A$. Checking the definition for A we see that there must hold $x = \frac{p}{q}$, $q > \varepsilon^{-1}$. This means for every such x, $|f(x)| < \varepsilon$. Thus ends the proof.