Math 117 Fall 2014 Homework 5

Due Thursday Oct. 16 3pm in Assignment Box

Question 1. (5 pts) *Let f, g: $\mathbb{R} \mapsto \mathbb{R}$ and $a \in \mathbb{R}$. Further assume $\lim_{x \to a} f(x) = L \in \mathbb{R}$ and $\lim_{x \to a} g(x) = M \in \mathbb{R}$.*

a) (2 pts) *Prove or disprove: Under the above assumptions, there is $M > 0$ such that $\forall x \in \mathbb{R}$, $|f(x)| < M$;*

b) (2 pts) *Prove **by definition**: $\lim_{x \to a} [f(x)\, g(x)] = L\, M$;*

c) (1 pt) *Compare your proof with that of $\lim_{n \to \infty} a_n\, b_n = a\, b$ in the lecture note for Oct.6. Is your proof simply a "translation" of the proof there? Are there any new ideas involved? Explain why these new ideas are necessary.*

Question 2. (5 pts) *Study $\lim_{x \to a} (\sqrt{x+1} - \sqrt{x})$ in the following situations:*

a) (2 pts) *$a = +\infty$;*

b) (3 pts) *$a = 0$;*

Justify any claim you make.

Question 3. (5 pts) *Let $\{a_n\}$ be a bounded sequence. Defined the set A to consist of all the a_n's, that is $A = \{a_1, a_2, a_3, \ldots\}$. Let $M := \sup A$. Prove that*

a) (1 pt) *$M \in \mathbb{R}$.*

b) (4 pts) *If there is no $n \in \mathbb{N}$ such that $M = a_n$, then there exists a increasing subsequence $\{a_{n_k}\}$ such that $\lim_{k \to \infty} a_{n_k} = M$. Make sure you check the definition for subsequences.*

Question 4. (5 pts) *Let $f: \mathbb{Q} \mapsto \mathbb{R}$ be defined as*

$$f(x) = \frac{1}{q} \qquad \text{when } x = \frac{p}{q},\ p, q \in \mathbb{Z}, q > 0, (p, q) = 1. \tag{1}$$

Let $a \in \mathbb{R}$. Study $\lim_{x \to a} f(x)$. You need to justify any claim you make.