Math 117 Fall 2014 Lecture 20 (Oct. 8, 2014)

Reading:

- In the following $a, b, L, M \in \mathbb{R}$. The cases of one or more of them are $+\infty$ or $-\infty$ are left as exercises. **Please make sure you work on these cases** – some of them may not be that straightforward – and compare your answers with those in the textbook (or any calculus books).

- Convergence implies boundedness.

  Lemma 1. *Let $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Then there is $M > 0$ such that for all $n \in \mathbb{N}$, $|a_n| < M$.*

  Proof. As $\lim_{n\to\infty} a_n = a$, there is $N \in \mathbb{N}$ such that for all $n \geqslant N$, $|a_n - a| < 1$. This implies

  $$\forall n \geqslant N, \qquad |a_n| = |a_n - a + a| \leqslant |a_n - a| + |a| < |a| + 1. \tag{1}$$

  Now define

  $$M := \max \left\{ |a| + 1, \max_{i=1,\ldots,N-1} (|a_i| + 1) \right\}. \tag{2}$$

  Then for every $n \in \mathbb{N}$, there are two cases:

  - $n \geqslant N$. We have

    $$|a_n| < |a| + 1 \leqslant M; \tag{3}$$

  - $n < N$. We have

    $$|a_n| \leqslant \max_{i=1,\ldots,N-1} |a_i| < \max_{i=1,\ldots,N-1} (|a_i| + 1) \leqslant M. \tag{4}$$

  Thus ends the proof. $\square$

  Exercise 1. Can we choose N according to $\forall n \geqslant N, \quad |a_n - a| < |a|$?

  Example 2. Consider the sequence $a_n = \frac{n+100}{n}$. Then if we apply the construction in the proof to this sequence, we can take $N = 101$, and have $M = 102$.

  Exercise 2. Try this on $a_n = e^{-n} \sin(n^2)$.

- $\div$.

  Theorem 3. *Let $\{a_n\}, \{b_n\}$ be sequences. Assume*

  i. $\lim_{n\to\infty} a_n = a \in \mathbb{R}$*;*

  ii. $\lim_{n\to\infty} b_n = b \in \mathbb{R}$*;*

  iii. $b \neq 0$.

  Then $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$.

  Proof. We first prove $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$.

  Let $\varepsilon > 0$. Let $N_1 \in \mathbb{N}$ be such that

  $$\forall n \geqslant N_1, \qquad |b_n - b| < \frac{|b|}{2}. \tag{5}$$

  Note that this is possible because $b \neq 0 \Longrightarrow \frac{|b|}{2} > 0$. Thus

  $$\forall n \geqslant N_1, \qquad |b_n| = |b - (b - b_n)| > |b| - |b - b_n| > \frac{|b|}{2}. \tag{6}$$

Next let $N_2 \in \mathbb{N}$ be such that

$$\forall n \geqslant N_2, \qquad |b_n - b| < \frac{|b|^2}{2}\,\varepsilon. \tag{7}$$

Finally set $N = \max\{N_1, N_2\}$. Then for every $n \geqslant N$,

$$\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b_n - b|}{|b_n| \cdot |b|} < \frac{|b|^2}{2}\,\varepsilon\,\frac{1}{|b|/2}\,\frac{1}{|b|} = \varepsilon. \tag{8}$$

Thus we have proved $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$.

Now as $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$, we have $\lim_{n\to\infty} a_n \cdot \frac{1}{b_n} = a \cdot \frac{1}{b} = \frac{a}{b}$. □

Remark 4. Note that the reason why we require $b \neq 0$ is **not** "if $b = 0$ the above proof would fail"

Exercise 3. Which step fails if $b = 0$?

Consider $a_n = 1$ and $b_n = \frac{1}{n}, \frac{-1}{n}, \frac{(-1)^n}{n}$. We see that $\lim_{n\to\infty} \frac{a_n}{b_n}$ is $+\infty, -\infty$, does not exist, repectively. Therefore if we allow $b = 0$, then the limit of $\frac{a_n}{b_n}$ cannot be obtained from knowing a, b only.

- Limit for rational functions.

Example 5. Prove $\lim_{n\to\infty} \frac{3\,n^3 + 2\,n}{7\,n^3 + 5\,n^2 + 1} = \frac{3}{7}$.

Proof. First we know that $\lim_{n\to\infty} n^{-1} = 0$. Then as $n^{-2} = n^{-1} \cdot n^{-1}$ application of the theorem for $\lim_{n\to\infty} a_n\, b_n$ we conclude $\lim_{n\to\infty} n^{-2} = 0$. Next as $n^{-3} = n^{-1} \cdot n^{-2}$ we reach $\lim_{n\to\infty} n^{-3} = 0$. Finally we know that if $c \in \mathbb{R}$, then $\lim_{n\to\infty} c = c$. This yields

$$\lim_{n\to\infty} 2\,n^{-2} = 0, \qquad \lim_{n\to\infty} 5\,n^{-1} = 0. \tag{9}$$

Summarizing the above, we have

$$\lim_{n\to\infty} (3 + 2\,n^{-2}) = 3, \qquad \lim_{n\to\infty} (7 + 5\,n^{-1} + n^{-3}) = 7. \tag{10}$$

As $7 \neq 0$, we have

$$\lim_{n\to\infty} \frac{3 + 2\,n^{-2}}{7 + 5\,n^{-1} + n^{-3}} = \frac{3}{7}. \tag{11}$$

But

$$\frac{3 + 2\,n^{-2}}{7 + 5\,n^{-1} + n^{-3}} = \frac{3\,n^3 + 2\,n}{7\,n^3 + 5\,n^2 + 1}, \tag{12}$$

Therefore $\lim_{n\to\infty} \frac{3\,n^3 + 2\,n}{7\,n^3 + 5\,n^2 + 1} = \frac{3}{7}$. □

- Some Typical Mistakes.

  The following are problematic proofs.

  - Prove $\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$.

    Proof. We have

$$\lim_{n\to\infty} \frac{(-1)^n}{n} = \lim_{n\to\infty} (-1)^n \cdot \lim_{n\to\infty} \frac{1}{n} = \left[\lim_{n\to\infty} (-1)^n\right] \cdot 0 = 0. \tag{13}$$

Thus ends the proof. □

Remark 6. $a\cdot 0=0$ for every real number a. But $\lim_{n\to\infty}(-1)^n$ is **not** a real number!

Exercise 4. Prove the limit by definition.

- Prove $\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})=0$.

Proof. We have

$$\lim_{n\to\infty}(\sqrt{n+1}-\sqrt{n})=\lim_{n\to\infty}\sqrt{n+1}-\lim_{n\to\infty}\sqrt{n}=\infty-\infty=0. \tag{14}$$

Thus ends the proof. □

Exercise 5. Find sequences $\{a_n\},\{b_n\}$ with $\lim_{n\to\infty}a_n=+\infty;\lim_{n\to\infty}b_n=+\infty$ for each of the following requirement:

- $\lim_{n\to\infty}[a_n-b_n]=1$;
- $\lim_{n\to\infty}[a_n-b_n]=+\infty$;
- $\lim_{n\to\infty}[a_n-b_n]=-\infty$;
- $\lim_{n\to\infty}[a_n-b_n]$ does not exist.