MATH 117 FALL 2014 LECTURE 19 (Oct. 6, 2014)

Reading:

- In the following $a, b, L, M \in \mathbb{R}$. The cases of one or more of them are $+\infty$ or $-\infty$ are left as exercises. Please make sure you work on these cases – some of them may not be that straightforward – and compare your answers with those in the textbook (or any calculus books).

- Comparison.

  THEOREM 1. *Let $\{a_n\}, \{b_n\}$ be sequences. Assume*

  i. $\lim_{n\to\infty} a_n = a$;

  ii. $\lim_{n\to\infty} b_n = b$;

  iii. $\forall n \in \mathbb{N}, a_n \leqslant b_n$.

  Then $a \leqslant b$.

  Proof. Assume the contrary, that is $a > b$. Then

  $$\lim_{n\to\infty} a_n = a \Longrightarrow \exists N_1 \in \mathbb{N}\ \forall n \geqslant N_1, \qquad |a_n - a| < \frac{a-b}{2}; \tag{1}$$

  $$\lim_{n\to\infty} b_n = b \Longrightarrow \exists N_2 \in \mathbb{N}\ \forall n \geqslant N_2, \qquad |b_n - b| < \frac{a-b}{2}. \tag{2}$$

  Thus for $n \geqslant N_1$,

  $$a_n - a \geqslant -|a_n - a| > \frac{b-a}{2} \Longrightarrow a_n > \frac{a+b}{2} \tag{3}$$

  and similarly for $n \geqslant N_2$,

  $$b_n - b \leqslant |b_n - b| < \frac{a-b}{2} \Longrightarrow b_n < \frac{a+b}{2}. \tag{4}$$

  Set $N = \max\{N_1, N_2\}$, then for any $n \geqslant N$, we have

  $$a_n > \frac{a+b}{2} > b_n. \tag{5}$$

  A contradiction to assumption iii. □

  Exercise 1. Try to prove the above theorem not using proof by contradiction.

  Exercise 2. Prove the following generalization:

  Let assumptions i, ii still hold. Replace iii with iii': $\exists N_0 \in \mathbb{N}\ \forall n \geqslant N_0, \quad a_n \leqslant b_n$. Then $a \leqslant b$.

  Exercise 3. Write down the corresponding "Comparison Theorem" for functions.

  Remark 2. Note that if we replace iii by $\forall n \in \mathbb{N}, \quad a_n < b_n$, the conclusion **does not** change to $a < b$, as can be seen from the example $a_n = \frac{1}{n}, b_n = \frac{2}{n}$.

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  THEOREM 3. *Let $\{a_n\}$, $\{b_n\}$ be sequences. Assume $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$. Then*

  $$\lim_{n\to\infty} (a_n \pm b_n) = a \pm b. \tag{6}$$

Proof. We prove the "+" case and leave the "-" case as exercise.

Let $\varepsilon > 0$ be arbitrary.

$$\lim_{n\rightarrow\infty} a_n = a \Longrightarrow \exists N_1 \in \mathbb{N}\ \forall n \geqslant N_1, \quad |a_n - a| < \frac{\varepsilon}{2}; \tag{7}$$

$$\lim_{n\rightarrow\infty} b_n = a \Longrightarrow \exists N_2 \in \mathbb{N}\ \forall n \geqslant N_2, \quad |b_n - b| < \frac{\varepsilon}{2}. \tag{8}$$

Now set $N = \max\{N_1, N_2\}$. Then for every $n \geqslant N$, we have

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leqslant |a_n - a| + |b_n - b| < \varepsilon. \tag{9}$$

Thus ends the proof. □

Exercise 4. Let $a_n = \frac{1}{n^2}, b_n = \frac{1}{n}$. Find the N_1, N_2, N as in the above proof.

Exercise 5. Write down and prove the corresponding theorem for functions.

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THEOREM 4. *Let $\{a_n\}, \{b_n\}$ be sequences. Assume $\lim_{n\rightarrow\infty} a_n = a, \lim_{n\rightarrow\infty} b_n = b$. Then*

$$\lim_{n\rightarrow\infty} (a_n\, b_n) = a\, b. \tag{10}$$

Proof. Let $\varepsilon > 0$ be arbitrary. We have the following lemma which will be proved in the next lecture:

LEMMA 5. *Let $\{a_n\}$ be a sequence and assume $\lim_{n\rightarrow\infty} a_n = a$. Then there is $M > 0$ such that $\forall n \in \mathbb{N},\ |a_n| < M$.*

Exercise 6. Does the lemma still hold if we allow a to be $+\infty$ or $-\infty$? Justify your answer.

Now we continue the proof. We have[1]

$$\lim_{n\rightarrow\infty} a_n = a \Longrightarrow \exists N_1 \in \mathbb{N}\ \forall n \geqslant N_1, \quad |a_n - a| < \frac{\varepsilon}{2\,(|b| + 1)}; \tag{11}$$

$$\lim_{n\rightarrow\infty} b_n = a \Longrightarrow \exists N_2 \in \mathbb{N}\ \forall n \geqslant N_2, \quad |b_n - b| < \frac{\varepsilon}{2\,M}. \tag{12}$$

Set $N = \max\{N_1, N_2\}$. Then for every $n \geqslant N$, we have

$$\begin{aligned} |a_n\, b_n - a\, b| &= |(a_n - a)\, b + a_n\, (b_n - b)| \\ &\leqslant |a_n - a|\, |b| + |a_n|\, |b_n - b| \\ &< \frac{\varepsilon}{2\,(|b| + 1)}\, |b| + M\, \frac{\varepsilon}{2\,M} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus ends the proof. □

Exercise 7. Let $a_n = \frac{3}{n^2}, b_n = \frac{2}{n}$. Find the M, N_1, N_2, N as in the above proof.

Exercise 8. Write down and prove the corresponding theorem for functions.

1. We use $|b| + 1$ instead of $|b|$ to make sure no "division by 0" will occur.