Math 117 Fall 2014 Lecture 18 (Oct. 3, 2014)

Reading:

- Some leftovers.
  - $a, L \in \mathbb{R}$. Definition for "$\lim_{x \to a} f(x) = L$" is not true.

$$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x \, 0 < |x - a| < \delta, \qquad |f(x) - L| \geqslant \varepsilon. \tag{1}$$

Remark 1. Note the difference between the above statement and "$\lim_{x \to a} f(x) \neq L$", which means "$\lim_{x \to a} f(x)$" exists but is a different number from L.

Exercise 1. Prove that if $\lim_{x \to a} f(x) = L$, then for any $L' \neq L$, "$\lim_{x \to a} f(x) = L'$ " is not true.

Exercise 2. Write down the definitions for "$\lim_{x \to a} f(x) = L$" is not true when a, L belong to the other eight cases.

Exercise 3. Prove that "$\lim_{x \to 0} \sin \frac{1}{x} = 0$" is not true.

Exercise 4. Write down the definition for $\lim_{x \to a} f(x)$ does not exist.

  - A few more questions about limit.

Exercise 5. Let $f: \mathbb{R} \mapsto \mathbb{R}$. Are the following two statements equivalent? Justify your answer.

$$\lim_{x \to 0} f(x) = L; \qquad \lim_{t \to +\infty} f\left(\frac{1}{t}\right) = L. \tag{2}$$

Problem 1. Let $f, g: \mathbb{R} \mapsto \mathbb{R}$ and $a, b, L \in \mathbb{R}$. Assume $\lim_{t \to a} f(t) = b$, $\lim_{x \to b} g(x) = L$. Prove or disprove:

There always holds $\lim_{t \to a} g(f(t)) = L$.

(Hint:[1])

Problem 2. Let $f, g: \mathbb{R} \mapsto \mathbb{R}$. Write down a reasonable definition for

$$\lim_{g(x) \to a} f(x) = L. \tag{3}$$

- In this lecture we discuss limits for a function $f: A \mapsto \mathbb{R}$ where $A \subset \mathbb{R}$. The definition of $\lim_{x \to a} f(x) = L$ may need to be modified.
- Simple cases.
  - $A = \mathbb{R} - \{a\}$.

Example 2. Study $\lim_{x \to 1} \frac{x^2 - 1}{x - 1}$.

Solution. Intuitively the limit is 2. Now we see whether the definition for $f: \mathbb{R} \mapsto \mathbb{R}$ applies to the current situation.

We try to check:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x \, 0 < |x - 1| < \delta \qquad \left| \frac{x^2 - 1}{x - 1} - 2 \right| < \varepsilon. \tag{4}$$

Let $\varepsilon > 0$ be arbitrary. Set $\delta = \varepsilon$. Then for every x satisfying $0 < |x - 1| < \delta$, we have

$$\left| \frac{x^2 - 1}{x - 1} - 2 \right| = |x + 1 - 2| = |x - 1| < \delta = \varepsilon. \tag{5}$$

Note that the first equality holds because we require $0 < |x - 1|$.

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1. No.

We see that the old definition still applies.

- $A \supseteq (c, d)$ where $a \in (c, d)$.

Example 3. Study $\lim_{x \to 1} \frac{1}{x^3}$.

Solution. Clearly the limit should be 1. The problem now is that $f(x) = \frac{1}{x^3}$ is not defined at $x = 0$. Let's see whether the old definition still applies.

Let $\varepsilon > 0$ be arbitrary. Set $\delta = \min\left\{\frac{\varepsilon}{38}, \frac{1}{2}\right\}$.[2] For every $0 < |x - 1| < \delta$ we have

$$\left|\frac{1}{x^3} - 1\right| = \left|\frac{x^3 - 1}{x^3}\right| = |x - 1| \left|\frac{x^2 + x + 1}{x^3}\right| < \delta \left|\frac{x^2 + x + 1}{x^3}\right| \leqslant \varepsilon. \tag{6}$$

Thus we see that the old definition still applies.

Exercise 6. If in the above proof we set $\delta = \min\left\{?, \frac{1}{3}\right\}$, what choice can we make to fill the "?"?

- Summary. Combining the above, we see that when there is (c, d) such that $a \in (c, d)$ and $(c, d) - \{a\} \subseteq A$, no change is needed in the definition for $\lim_{x \to a} f(x) = L$.

- More complicated cases.

  For more complicated A the definition for $\lim_{x \to a} f(x) = L$ needs to be revised.

Example 4. Let $f(x) = \sqrt{x(1 - x)}$. Study $\lim_{x \to 2} f(x)$.

Solution. This is a wrong question to ask, as the idea for limit is "as x approaches a, does f approach L?" In this example the domain of f is $[0, 1]$ and it is not possible for x to approach a.

DEFINITION 5. (LIMIT POINT) *Let $A \subseteq \mathbb{R}, a \in \mathbb{R}$. a is said to be a "limit point" (or "cluster point") of the set A if and only if*

$$\forall \delta > 0 \ \exists a' \in A \qquad 0 < |a - a'| < \delta. \tag{7}$$

Example 6. Let $A = \left\{\frac{1}{n} | \, n \in \mathbb{N}\right\}$. Find the set of its limit points.

Solution. We claim that this set consists of the single point 0, that is it is $\{0\}$. To prove this we need to justify two things.

- 0 is a limit point of A.

  Let $\delta > 0$ be arbitrary. Take $n \in \mathbb{N}$ such that $n > \delta^{-1}$. Then set $a' = \frac{1}{n}$. We see that $0 \neq a'$ which gives $|0 - a'| > 0$, and furthermore $|0 - a'| = \frac{1}{n} < \delta$.

- Let $a \in \mathbb{R}, a \neq 0$, then a is not a limit point of A.

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2. To understand this choice of δ, we need to first clearly understand the requirements on δ. The choice of δ should be such that

i. When $0 < |x - 1| < \delta$, $x \neq 0$;

ii. When $0 < |x - 1| < \delta$, there is a number $M \in \mathbb{R}$ such that $|x^2 + x + 1| \leqslant M$;

iii. When $0 < |x - 1| < \delta$, there is a number $m > 0$ such that $|x^3| > m$ – otherwise $\frac{1}{|x^3|}$ can get arbitrarily large;

iv. When $0 < |x - 1| < \delta$, $\left|\frac{1}{x^3} - 1\right| < \varepsilon$.

The first is satisfied when $\delta \leqslant 1$, the second when δ is finite (which is automatically satisfied for any δ we may choose), the third when $\delta < 1$. As usual, we will deal with the fourth requirement after making a specific choice of δ satisfying all of i – iii. Let's say we pick $\delta = \frac{1}{2}$. Then we have $\left|\frac{1}{x^3} - 1\right| < 38\delta$ and the choice for iv is now obvious.

There are three cases.

- $a > 1$. Set $\delta = a - 1 > 0$. We see that $\forall a' \in A$, there holds $a' \leqslant 1$ and consequently $|a - a'| \geqslant \delta$. Therefore a cannot be a limit point of A;
- $a < 0$. Set $\delta = |a| > 0$. We see that $\forall a' \in A$, there holds $a' > 0$ and consequently $|a - a'| > |a| = \delta$. Therefore a cannot be a limit point of A;
- $a \in [0, 1]$. This case is a bit tricky. We define a set $A_a := \left\{ n \in \mathbb{N} | \frac{a}{2} < \frac{1}{n} < \frac{3a}{2} \right\}$. Then A_a is a finite set. Now set

  $$\delta := \min \left\{ \frac{a}{2}, \min_{a' \in A_a, a' \neq a} \{|a' - a|\} \right\}. \tag{8}$$

  Then $\delta > 0$. Furthermore for every $a' \in A$, if $a' \notin A_a$, we have

  $$|a - a'| \geqslant \frac{a}{2} \geqslant \delta; \tag{9}$$

  On the other hand if $a' \in A_a$, we have

  $$|a - a'| \geqslant \min_{a' \in A_a, a' \neq a} \{|a' - a|\} \geqslant \delta. \tag{10}$$

  Thus for every $a' \in A$ we have $|a - a'| \geqslant \delta$ and consequently a is not a limit point of A.

Exercise 7. Calculate the set of limit points for $A := (0, 1)$. Justify.

Exercise 8. Calculate the set of limit points for $A = \mathbb{Q}$. Justify.

Exercise 9. Calculate the set of limit points for $\mathbb{N}$. Justify.

Problem 3. Calculate the set of limit points for $A := \left\{ \frac{1}{m^2} + \frac{1}{n} | m, n \in \mathbb{N} \right\}$. Justify your answer.

Problem 4. Let $A \subseteq \mathbb{R}$. Let A' be the set of limit points of A. Let A'' be the set of limit points of A'. Find the relation between A' and A'', then justify.

- A definition that is universally applicable.

  Let $f: A \mapsto \mathbb{R}$. Let $a \in \mathbb{R}$ be a limit point of A. We say $\lim_{x \to a} f(x) = L$ for $L \in \mathbb{R}$ if and only if

  $$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \text{ satisfying } 0 < |x - a| < \delta \text{ and } x \in A, \qquad |f(x) - L| < \varepsilon. \tag{11}$$

Exercise 10. Let $f(x): \mathbb{Q} \mapsto \mathbb{R}$ be defined as $f(x) = x$. Let $a \in \mathbb{R}$. Prove $\lim_{x \to a} f(x) = a$.

Exercise 11. Revise the definition for the case $L = +\infty$.

Exercise 12. Revise the definition for the case $a = -\infty$. Do you have any difficulty doing so?

Problem 5. Let $f: \mathbb{Q} \mapsto \mathbb{R}$ be defined as

$$f(x) = \frac{1}{q} \qquad \text{when } x = \frac{p}{q} \text{ where } p, q \in \mathbb{Z}, q > 0, (p, q) = 1. \tag{12}$$

Let $a \in \mathbb{R}$. Study $\lim_{x \to a} f(x)$.