Math 117 Fall 2014 Homework 4 Solutions

Due Thursday Oct. 9 3pm in Assignment Box

Question 1. (10 pts) *Prove the following statements* ***by definition***.

a) (2 pts) $\lim_{n\to\infty}\frac{n!}{n^n}=0$.

b) (2 pts) $\lim_{n\to\infty}\left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right]=1$.

c) (2 pts) *The sequence* $\left\{(-1)^{n^2}\right\}$ *is divergent.*

d) (2 pts) $\lim_{x\to 0}x^2\sin\left(\frac{1}{x}\right)=0$.

e) (2 pts) *The limit* $\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ *does not exist.*

Proof.

a) Let $\varepsilon>0$ be arbitrary. Set $N>-2\log_2\varepsilon$, then we have for all $n\geqslant N$,

$$\left|\frac{n!}{n^n}-0\right|=\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{1}{n}\leqslant\frac{N/2}{n}\cdot\frac{(N/2)-1}{n}\cdots\frac{1}{n}\leqslant\left(\frac{1}{2}\right)^{N/2}=\left(\frac{1}{2}\right)^{-\log_2\varepsilon}<\varepsilon. \tag{1}$$

Thus ends the proof.

b) First we observe that for every $n\in\mathbb{N}$,

$$\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}<\frac{1}{\sqrt{n^2}}+\cdots+\frac{1}{\sqrt{n^2}}=1. \tag{2}$$

Let $\varepsilon>0$ be arbitrary. Set $N>\varepsilon^{-1}$, then we have for all $n\geqslant N$,

$$\begin{aligned}\left|\frac{1}{\sqrt{n^2+1}}+\cdots+\frac{1}{\sqrt{n^2+n}}-1\right| &= 1-\left(\frac{1}{\sqrt{n^2+1}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right)\\ &< 1-\left(\frac{1}{\sqrt{n^2+2n+1}}+\cdots+\frac{1}{\sqrt{n^2+2n+1}}\right)\\ &= 1-\frac{n}{n+1}=\frac{1}{n+1}<\frac{1}{N}<\varepsilon.\end{aligned} \tag{3}$$

Thus ends the proof.

c) Let $a\in\mathbb{R}$ be arbitrary. We prove that $\lim_{n\to\infty}(-1)^{n^2}=a$ cannot hold. Assume the contrary. Then there is $N\in\mathbb{N}$ such that for all $n\geqslant N$,

$$|a_n-a|<1. \tag{4}$$

Now we discuss two cases.

- $a\geqslant 0$. Let $n=2N+1$. Then we have $|a_n-a|=|(-1)-a|=1+|a|\geqslant 1$. Contradiction.
- $a<0$. Let $n=2N$. Then we have $|a_n-a|=|1-a|=1+|a|>1$. Contradiction.

d) Let $\varepsilon > 0$ be arbitrary. Take $\delta = \varepsilon^{1/2}$. Then for every $0 < |x - 0| < \delta$ we have

$$\left| x^2 \sin\frac{1}{x} - 0 \right| = |x|^2 \left| \sin\frac{1}{x} \right| \leqslant |x|^2 < \delta^2 = \varepsilon. \tag{5}$$

e) Let $L \in \mathbb{R}$ be arbitrary. We prove the $\lim_{x \to 0} \sin\left(\frac{1}{x}\right) = L$ cannot hold. Assume the contrary. Then there is $\delta > 0$ such that for all $0 < |x| < \delta$, $\left|\sin\frac{1}{x} - L\right| < 1$. We discuss two cases.

- $L \geqslant 0$. Let $n \in \mathbb{N}$ be such that $n > \delta^{-1}$. We take $x = \left(2 n \pi + \frac{3 \pi}{2}\right)^{-1} < n^{-1} < \delta$. Then

$$\left| \sin\frac{1}{x} - L \right| = |-1 - L| = 1 + L \geqslant 1, \tag{6}$$

  contradiction;

- $L < 0$. Let $n \in \mathbb{N}$ be such that $n > \delta^{-1}$. We take $x = \left(2 n \pi + \frac{\pi}{2}\right)^{-1} < n^{-1} < \delta$. Then

$$\left| \sin\frac{1}{x} - L \right| = |1 - L| = 1 + |L| > 1, \tag{7}$$

  contradiction again. □

Question 2. (5 pts) *A sequence $\{a_n\}$ is said to be "bounded above" if and only if there is $M > 0$ such that $\forall n \in \mathbb{N}$, $a_n \leqslant M$.*

a) (2 pts) *Write down the definition of "$\{a_n\}$ is not bounded above", that is write down the working negation of "$\{a_n\}$ is bounded above".*

b) (3 pts) *Prove or disprove the following statement:*

If $\{a_n\}$ is not bounded above, then $\lim_{n \to \infty} a_n = +\infty$.

Solution.

a) $\forall M > 0 \; \exists n \in \mathbb{N}, \qquad a_n > M.$

b) Let $a_n = [1 + (-1)^n]\, n$, $n \in \mathbb{N}$.

- $\{a_n\}$ is not bounded above.

  Let $M > 0$ be arbitrary. Take $n \in \mathbb{N}$ such that $n > M$ and is even. Then we have

$$a_n = 2 n > M. \tag{8}$$

  Therefore $\{a_n\}$ is not bounded above.

- $\lim_{n \to \infty} a_n = +\infty$ is not true.

  To prove this we need to show

$$\exists M > 0 \; \forall N \in \mathbb{N} \; \exists n \geqslant N, \qquad a_n \leqslant M. \tag{9}$$

  Take $M = 1$. Let $N \in \mathbb{N}$ be arbitrary. Take $n = 2 N + 1 \geqslant N$. Then we have

$$a_n = [1 + (-1)^{2N+1}]\, n = 0 \leqslant 1 = M. \tag{10}$$

  Thus ends the proof.

Question 3. (5 pts) *Let $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n} = \sum_{k=1}^{n} \frac{1}{k}$. Prove by definition that $\lim_{n \to \infty} H_n = +\infty$.*

Proof. Let $M > 0$ be arbitrary. Set $N > 2^{2M}$. Then we have, for all $n \geqslant N$,

$$\begin{aligned} H_n &> 1 + \frac{1}{2} + \cdots + \frac{1}{2^{2M} - 1} & (11)\\ &= 1 + \left(\frac{1}{2^1} + \frac{1}{2^2 - 1} \right) + \left(\frac{1}{2^2} + \cdots + \frac{1}{2^3 - 1} \right) + \cdots + \left(\frac{1}{2^{2M-1}} + \cdots + \frac{1}{2^{2M} - 1} \right) & (12)\\ &> 1 + \frac{2^1}{2^2} + \frac{2^2}{2^3} + \cdots + \frac{2^{2M-1}}{2^{2M}} & (13)\\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{2} \qquad \left(2M - 1 \ \ \frac{1}{2}\text{'s} \right) & (14)\\ &> \frac{2M}{2} = M. & (15) \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} H_n = +\infty$ by definition. □