MATH 117 FALL 2014 LECTURE 17 (OCT. 2, 2014)

Reading: Bowman: §3.C.

- In this lecture we discuss limits for a function $f: \mathbb{R} \mapsto \mathbb{R}$.
- Definition of $\lim_{x \to a} f(x) = L$ for $a, L \in \mathbb{R}$.

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x 0 < |x - a| < \delta, \qquad |f(x) - L| < \varepsilon. \tag{1}$$

Example 1. Prove $\lim_{x \to 0} x^3 = 0$ by definition.

Proof. Let $\varepsilon > 0$ be arbitrary. Set $0 < \delta < \varepsilon^{1/3}$. Then $\forall x 0 < |x - a| < \delta$

$$|x^3 - 0| = |x|^3 < \delta^3 < \varepsilon. \tag{2}$$

This ends the proof. □

Example 2. Prove $\lim_{x \to 1} x^3 = 1$ by definition.

Proof. Let $\varepsilon > 0$ be arbitrary. Set $0 < \delta < \min \left\{ \frac{\varepsilon}{7}, 1 \right\}$. Then for every x satisfying $0 < |x - 1| < \delta$ we have, by triangle inequality, $|x| < 1 + \delta < 1 + 1 = 2$. Now for such x we calculate

$$|x^3 - 1| = |x - 1| \, |x^2 + x + 1| < \delta \, [|x|^2 + |x| + 1] < 7 \, \delta < 7 \cdot \frac{\varepsilon}{7} = \varepsilon. \tag{3}$$

Thus ends the proof. □

Exercise 1. Let $n \in \mathbb{N}$, $a \in \mathbb{R}$. Prove $\lim_{x \to a} x^n = a^n$ by definition.

Exercise 2. Let $P(x)$ be a polynomial and $a \in \mathbb{R}$. Prove $\lim_{x \to a} P(x) = P(a)$ by definition.

- $a, L = \pm\infty$?
  - As a, L can each be a real number or $\pm\infty$, we have nine cases in total.
  - Definition for $\lim_{x \to +\infty} f(x) = L \in \mathbb{R}$.

$$\forall \varepsilon > 0 \; \exists R > 0 \; \forall x > R \qquad |f(x) - L| < \varepsilon. \tag{4}$$

  - Definition for $\lim_{x \to a \in \mathbb{R}} f(x) = -\infty$.

$$\forall M < 0 \; \exists \delta > 0 \; \forall x \; 0 < |x - a| < \delta, \qquad f(x) < M. \tag{5}$$

Exercise 3. Write down the definitions for the remaining six cases.

Exercise 4. Prove $\lim_{x \to -\infty} x^3 + x^2 = -\infty$ by definition.