Math 117 Fall 2014 Lecture 16 (Oct. 1, 2014)

Reading:

- Recall definition of limit: Let $a \in \mathbb{R}$, $\{a_n\}$ a sequence of real numbers. Say $\lim_{n\to\infty} a_n = a$ if and only if

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geqslant N, \qquad |a_n - a| < \varepsilon. \tag{1}$$

Example 1. Prove $\lim_{n\to\infty} 2^{-n} = 0$.

Proof. Let $\varepsilon > 0$ be arbitrary. Set $N > -\log_2 \varepsilon$. Then for every $n \geqslant N$ we have

$$|2^{-n} - 0| = 2^{-n} \leqslant 2^{-N} < \varepsilon. \tag{2}$$

Thus ends the proof. □

Exercise 1. Prove $\lim_{n\to\infty} e^{-n} = 0$ by definition. Find the "N" for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$.

Exercise 2. Prove $\lim_{n\to\infty} \frac{\sqrt{n} - 1}{\sqrt{n}} = 1$ by definition. Find the "N" for $\varepsilon = 10^{-2}, 10^{-3}, 10^{-4}$.

Exercise 3. Let $\lim_{n\to\infty} a_n = a \in \mathbb{R}$. Prove by definition that $\lim_{n\to\infty} a_{n+2} = a$.

Problem 1. Prove $\lim_{n\to\infty} n^{1/n} = 1$ by definition. (Hint:[1])

- How to prove "$\lim_{n\to\infty} a_n = a$ is not true"?
  - Working negation of $\lim_{n\to\infty} a_n = a$:

$$\exists \varepsilon_0 > 0 \ \forall N \in \mathbb{N} \ \exists n_0 \geqslant N \qquad |a_{n_0} - a| \geqslant \varepsilon_0. \tag{3}$$

  - To prove, first find out the appropriate value of ε_0, then prove that for this particular ε_0 there holds $\forall N \in \mathbb{N} \ \exists n \geqslant N \qquad |a_n - a| \geqslant \varepsilon_0$.

Theorem 2. *If $\lim_{n\to\infty} a_n = a$, then there is no $b \neq a$ such that $\lim_{n\to\infty} a_n = b$.*

Proof. Set $\varepsilon_0 = \frac{|b-a|}{2}$. Let $N \in \mathbb{N}$ be arbitrary. Now we find $n \geqslant N$ such that $|a_n - b| \geqslant \varepsilon_0$.

As $\lim_{n\to\infty} a_n = a$, there is $N_1 \in \mathbb{N}$ such that $\forall n \geqslant N_1$, $|a_n - a| < \varepsilon_0$. We choose one $n_0 > \max\{N, N_1\}$. Then for this n_0

$$\begin{aligned} |a_{n_0} - b| &= |(a-b) - (a - a_{n_0})| && (4)\\ &\geqslant |a-b| - |a - a_{n_0}| && (5)\\ &> |a-b| - \varepsilon_0 = \varepsilon_0. && (6) \end{aligned}$$

Thus ends the proof. □

Exercise 4. Prove that $\lim_{n\to\infty} e^{-n} \sin n = 1$ does not hold.

- Convergence/Divergence.
  - Let $\{a_n\}$ be a sequence. If there is $a \in \mathbb{R}$ such that $\lim_{n\to\infty} a_n = a$, then we say $\{a_n\}$ converges (is convergent). Otherwise we say $\{a_n\}$ diverges (is divergent).

    More rigorously, $\{a_n\}$ is convergent if and only if

$$\exists a \in \mathbb{R} \ \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n \geqslant N \qquad |a_n - a| < \varepsilon; \tag{7}$$

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1. Set $b_n := \sqrt{n^{1/n}}$ and $h_n := b_n - 1$. Then $n\,h_n < 1 + n\,h_n \leqslant (1+h_n)^n = n^{1/2}$. Therefore $h_n < n^{-1/2}$.

$\{a_n\}$ is divergent if and only if

$$\forall a \in \mathbb{R}\ \exists \varepsilon > 0\ \forall N \in \mathbb{N}\ \exists n \geqslant N \qquad |a_n - a| \geqslant \varepsilon. \tag{8}$$

∘

Example 3. Prove that $\{(-1)^n\}$ is divergent.

Proof. Let $a \in \mathbb{R}$ be arbitrary. Take $\varepsilon = 1$. Let $N \in \mathbb{N}$ be arbitrary. There are two cases.

- $a \geqslant 0$.
  Take $n = 2N + 1 \geqslant N$. We have $|a_n - a| = |-1 - a| = 1 + a \geqslant 1 = \varepsilon$.
- $a < 0$.
  Take $n = 2N \geqslant N$. We have $|a_n - a| = |1 - a| = 1 + |a| \geqslant 1 = \varepsilon$.

Thus ends the proof. □

Example 4. Prove that $\{n\}$ is divergent.

Proof. Let $a \in \mathbb{R}$ be arbitrary. Take $\varepsilon = 1$. Let $N \in \mathbb{N}$ be arbitrary. Take $n = \max\{N, \lceil |a| \rceil + 1\} \geqslant N$ where the "ceiling function" $\lceil x \rceil$ denotes the smallest integer no less than x. Now we have

$$|a_n - a| = |n - a| \geqslant \lceil |a| \rceil + 1 - |a| \geqslant 1. \tag{9}$$

Thus ends the proof. □

Exercise 5. Prove by definition that $\{e^{-1/n}(-1)^n\}$ is divergent.

Problem 2. Prove by defintion that $\{\sin n\}$ is divergent. (Hint:[2])

∘ Divergence to $\pm\infty$.

Definition 5. *A sequence $\{a_n\}$ is said to diverge to $+\infty$, denoted $\lim_{n\to\infty} a_n = +\infty$, if and only if*

$$\forall M > 0\ \exists N \in \mathbb{N}\ \forall n \geqslant N \qquad a_n > M. \tag{10}$$

Exercise 6. Define $\lim_{n\to\infty} a_n = -\infty$.

Exercise 7. Prove that $\lim_{n\to\infty} a_n = +\infty$ if and only if

$$\forall M \in \mathbb{R}\ \exists N \in \mathbb{N}\ \forall n \geqslant N \qquad a_n > M. \tag{11}$$

Exercise 8. Let $a_n = e^{n^2}$. Find $N \in \mathbb{N}$ such that $\forall n \geqslant N$, $a_n > M$ for $M = 10^2, 10^3, 10^4$.

Example 6. Prove $\lim_{n\to\infty} e^n = +\infty$.

Proof. Let $M > 0$ be arbitrary. Set $N > \ln M$. Then for every $n \geqslant N$ we have

$$e^n \geqslant e^N > M. \tag{12}$$

Thus ends the proof. □

Example 7. Let $H_n := 1 + \frac{1}{2} + \cdots + \frac{1}{n}$. Prove $\lim_{n\to\infty} H_n = +\infty$.

2. For $x > 0$ denote by $\{x\}$ the remainder of $x \div (2\pi)$, that is $\{x\} \in [0, 2\pi)$ and $x = 2\pi k + \{x\}$ for some $k \in \mathbb{Z}$. Now prove that for every $\delta > 0$ there is $n \in \mathbb{N}$ such that $\{n\} < \delta$.

Idea. The following argument is due to middle age polymath Nicole Oresme (1320 – 1382):

$$\begin{aligned} 1+\frac{1}{2}+\cdots &= 1+\left(\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{4}+\cdots+\frac{1}{7}\right)+\left(\frac{1}{8}+\cdots+\frac{1}{15}\right)+\cdots \\ &> \frac{1}{2}+2\times\frac{1}{4}+4\times\frac{1}{8}+8\times\frac{1}{16}+\cdots \\ &= \frac{1}{2}+\frac{1}{2}+\cdots=\frac{+\infty}{2}=+\infty. \end{aligned} \tag{13}$$

Exercise 9. The above is **NOT** a proof by our modern standard. Turn this idea into a rigorous proof (by definition) of $\lim_{n\to\infty} H_n = +\infty$.

Problem 3. Apply the same idea to study $\lim_{n\to\infty} H_{n,p}$ where

$$H_{n,p} := 1+\frac{1}{2^p}+\frac{1}{3^p}+\cdots \tag{14}$$

with $p>0$.

- More exercises.

  In all of the following you should prove by definition.

  Exercise 10. Prove $\lim_{n\to\infty} \frac{n!}{n^n} = 0$.

  Exercise 11. Prove $\lim_{n\to\infty} \left[\frac{1}{n^2}+\frac{1}{(n+1)^2}+\cdots+\frac{1}{(n+n)^2}\right] = 0$.

  Exercise 12. Prove $\lim_{n\to\infty} \left[\frac{1}{\sqrt{n}}+\frac{1}{\sqrt{n+1}}+\cdots+\frac{1}{\sqrt{n+n}}\right] = +\infty$.

  Exercise 13. Prove $\lim_{n\to\infty} \left[\frac{1}{\sqrt{n^2+1}}+\frac{1}{\sqrt{n^2+2}}+\cdots+\frac{1}{\sqrt{n^2+n}}\right] = 1$.

  Exercise 14. Let $a, b>0$. Prove $\lim_{n\to\infty} (a^n+b^n)^{1/n} = \max\{a, b\}$.

  Exercise 15. For $n\in\mathbb{N}$ let $\nu(n)$ be the number of distinct prime factors of n. For example $\nu(12)=2$. Prove $\lim_{n\to\infty} \frac{\nu(n)}{n} = 0$.

  Problem 4. Let $\nu(n)$ be defined as in the above exercise. Let $a\geqslant 0$. Study $\lim_{n\to\infty} \frac{\nu(n)}{n^a}$.

  Problem 5. The Dirichlet function is defined as

  $$D(x) = \begin{cases} 1 & x\in\mathbb{Q} \\ 0 & x\notin\mathbb{Q} \end{cases}. \tag{15}$$

  Prove that

  $$D(x) = \lim_{n\to\infty}\left[\lim_{m\to\infty} (\cos(n!\,\pi x))^{2m}\right]. \tag{16}$$