Math 117 Fall 2014 Lecture 14 (Sept. 25, 2014)

Reading: 314 Proof and Logic: §2; 314 Midterm Review §A, §D.

- Mathematical statement:
  A statement that is either true or false, but not both.
- Propositional Logic: Statements and their combinations, no variable involved.
  - Truth value.
    If a statement A is true, we say $A = T$; Otherwise A must be false and we say $A = F$.
  - Conjunction.
    - $A \wedge B$ is true if and only if both A, B are true. For example $(3 = 1) \wedge \left(\sqrt{5} \notin \mathbb{Q}\right)$ is false while $(3 \neq 1) \wedge \left(\sqrt{5} \notin \mathbb{Q}\right)$ is true.
    - Reads: "A and B".
  - Disjunction.
    - $A \vee B$ is false if and only if both A, B are false. Thus $(3 = 1) \vee \left(\sqrt{5} \notin \mathbb{Q}\right)$ is true but $(3 = 1) \vee \left(\sqrt{5} \in \mathbb{Q}\right)$ is false.
    - Reads: "A or B".
  - Negation.
    - $\neg A$ is true if and only if A is false. Thus $\neg(3 = 1)$ is true while $\neg\left(\sqrt{5} \notin \mathbb{Q}\right)$ is false.
    - Reads: "Not A".
  - Conditional.
    - $A \Longrightarrow B$ is false if and only if A is true and B is false.
    - Reads: "A implies B", "If A then B", "B if A", "A only if B", "A is sufficient for B", "B is necessary for A".
  - Bi-conditional.
    - $A \Longleftrightarrow B$ is defined as
      $$(A \Longrightarrow B) \wedge (B \Longrightarrow A). \tag{1}$$
  - Truth table.
    Any statement in propositional logic is the result of combining finitely many, say m, "atom" statements through $\wedge, \vee, \neg, \Longrightarrow, \Longleftrightarrow$. As each "atom" statement can only take true or false, we see that there are only 2^m possible situations. Therefore all the proofs in propositional logic can be done with the "truth table", where every possible truth value assignment to the m "atom" statements are simply listed.

    Example 1. Prove that $A \Longrightarrow B$ is equivalent to $\neg A \vee B$.

    Proof. We list all possible cases.

$$\begin{array}{llll} A & B & A \Longrightarrow B & \neg A \vee B \\ T & F & F & F \\ F & T & T & T \\ T & T & T & T \\ F & F & T & T \end{array}. \tag{2}$$

Thus the proof ends. $\square$

- Predicative logic.
  - Predicative logic introduces variables into statements. For example,
$$\forall x \exists y \quad y = x^2. \tag{3}$$
    Reads: "For every x there is y such that $y = x^2$."
  - $\forall$: For every, for all.
    - A shorthand of conjunction of any number (could be infinite) of statements.
$$\forall x \in \mathbb{N}, \qquad x \geqslant 1 \tag{4}$$
      is a shorthand for
$$(1 \geqslant 1) \wedge (2 \geqslant 1) \wedge (3 \geqslant 1) \cdots \tag{5}$$
  - $\exists$: There exists.
    - A shorthand of disjunction.
$$\exists x \in \mathbb{N}, \qquad x \geqslant 1 \tag{6}$$
      is a shorthand for
$$(1 \geqslant 1) \vee (2 \geqslant 1) \vee (3 \geqslant 1) \vee \cdots \tag{7}$$
  - The order is important.
    For example $\forall x \exists y \quad y = x^2$ is true while $\exists y \forall x \quad y = x^2$ is false.
  - Working negation.
    - Often (for example when doing proof by contradiction) we need to use the negation of a certain statement, say A. Usually, simply writing down the negative statement $\neg A$ is not helpful. It is necessary to find another positive statement B that is equivalent to $\neg A$. This B is called the "working negation" of A.
    - For example, when we set up proof by contradiction for $\sqrt{2} \notin \mathbb{Q}$, we do not start with the negative statement $\neg\left(\sqrt{2} \notin \mathbb{Q}\right)$ – if we do we would go nowhere – but with the positive "working negation" $\sqrt{2} \in \mathbb{Q}$.
    - Rules for obtaining working negation: $\forall \longleftrightarrow \exists$. Thus the working negation of
$$\forall x \exists y \forall z \quad P(x, y, z) \tag{8}$$
      is
$$\exists x \forall y \exists z \qquad \neg P(x, y, z). \tag{9}$$

      Example 2. Let $A \subseteq \mathbb{R}$. A function $f: A \mapsto \mathbb{R}$ is uniformly continuous on A if and only if
$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall x, y \in A \text{ satisfying } |x - y| < \delta \qquad |f(x) - f(y)| < \varepsilon. \tag{10}$$
      Then a function f that is **not** uniformly continuous on A is characterized by the working negation of (10):
$$\exists \varepsilon > 0 \ \forall \delta > 0 \ \exists x, y \in A \text{ satisfying } |x - y| < \delta \qquad |f(x) - f(y)| \geqslant \varepsilon. \tag{11}$$

Note. Please make sure you understand why the red parts in the above example stays unchanged.

Exercise 1. A function f is continuous at x_0 if and only if

$$\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \text{ satisfying } |x - x_0| < \delta \qquad |f(x) - f(x_0)| < \varepsilon. \tag{12}$$

Characterize a function that is not continuous at x_0.

Exercise 2. A function $f: A \mapsto \mathbb{R}$ is bounded if and only if

$$\exists M > 0 \; \forall x \in A \qquad |f(x)| \leqslant M. \tag{13}$$

Characterize a unbounded function.

Exercise 3. A function $f: \mathbb{R} \mapsto \mathbb{R}$ is monotone if and only if it is either increasing or decreasing. Characterize a function that is no monotone.