# Math 117 Fall 2014 Lecture 13 (Sept. 24, 2014)

**Reading:** Dr. Bowman's book: §1.E, §1.F.

- Induction.
  - To prove that infinitely many statements are true. These infinitely many statements must be ordered through a parameter $n \in \mathbb{N}$.[1] That is these statements can be listed as $P(1), P(2), P(3), \ldots$ For example, the claim "$2^{2^n}+1$ is prime for every $n \in \mathbb{N}$" is a list of infinitely many statements:

$$\begin{aligned} P(1) &: 2^{2^1}+1 \text{ is prime;} \\ P(2) &: 2^{2^2}+1 \text{ is prime;} \\ P(3) &: 2^{2^3}+1 \text{ is prime;} \\ &\vdots \quad \vdots \end{aligned}$$

  - Two steps.
    - Show that $P(1)$ is true;
    - Show that, if $P(k)$ is true then $P(k+1)$ is true.
  - Why does it work?

    That induction works on $\mathbb{N}$ is in fact a axiom on the set of natural numbers:

    Axiom. The set $\mathbb{N}$ has the following property.
    For any $S \subseteq \mathbb{N}$, if

    1. $1 \in S$, and
    2. Once $k \in S$ there must hold $k+1 \in S$,

    then $S = \mathbb{N}$.

**Example 1.** Prove that $1^3+2^3+\cdots+n^3 = \left(\frac{n\,(n+1)}{2}\right)^2$.

**Proof.** Denote the statements by $P(n)$, that is let $P(1)$: $1^3 = \left(\frac{1\,(1+1)}{2}\right)^2$, $P(2)$: $1^3+2^3 = \left(\frac{2\,(2+1)}{2}\right)^2$, and so on. We check

- $P(1)$ is true. This is obvious as $\left(\frac{1\,(1+1)}{2}\right)^2 = 1$.
- If $P(k)$ is true then so is $P(k+1)$.

  Since $P(k)$ is true, we have

$$1^3+2^3+\cdots+k^3 = \left(\frac{k\,(k+1)}{2}\right)^2. \tag{1}$$

1. Of course two parameters $m, n \in \mathbb{N}$ is also OK.

Adding $(k+1)^3$ to both sides we have

$$\begin{aligned} 1^3+2^3+\cdots+k^3+(k+1)^3 &= \left(\frac{k(k+1)}{2}\right)^2+(k+1)^3 \\ &= \left(\frac{k}{2}\right)^2(k+1)^2+(k+1)^3 \\ &= \left[\left(\frac{k}{2}\right)^2+k+1\right](k+1)^2 \\ &= (k^2+4k+4)\frac{(k+1)^2}{4} \\ &= \left(\frac{(k+1)(k+2)}{2}\right)^2 \\ &= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2. \end{aligned} \tag{2}$$

we see that $P(k+1)$ must hold and the proof ends. □

**Example 2.** Prove that $\sqrt{5}$, $\sqrt{5\sqrt{5}}$, ... are all strictly less than 5.

**Proof.** Denote $a_n := \sqrt{5\sqrt{5\sqrt{\cdots\sqrt{5}}}}$ ($n$ square roots). Then we need to prove that all of the $P(n)$: $a_n < 5$ are true.

- $P(1)$ is true. Clearly $a_1 = \sqrt{5} < 5$.
- If $P(k)$ is true then so is $P(k+1)$. Assume $a_k < 5$. Then

$$a_{k+1} = \sqrt{5\,a_k} < \sqrt{5\cdot 5} = 5. \tag{3}$$

Thus ends the proof. □

**Exercise 1.** Let $x \neq 1$. Prove

$$1+x+x^2+\cdots+x^n = \frac{1-x^{n+1}}{1-x} \tag{4}$$

for all $n \in \mathbb{N}$.

**Exercise 2.** Let $n \geqslant 5$. Prove

$$2^n > n^2. \tag{5}$$

**Problem 1.** Prove the following. Let $n \in \mathbb{N}$, $x \neq k\pi$ for any $k \in \mathbb{Z}$.

$$\sin x+\sin 2x+\cdots+\sin(nx) = \frac{\sin\left(\frac{n+1}{2}x\right)\sin\left(\frac{nx}{2}\right)}{\sin\left(\frac{x}{2}\right)}; \tag{6}$$

$$\frac{1}{2}+\cos x+\cos 2x+\cdots+\cos nx = \frac{\sin\left(\left(n+\frac{1}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)}. \tag{7}$$

$$\frac{1}{2}\tan\left(\frac{x}{2}\right)+\frac{1}{4}\tan\left(\frac{x}{4}\right)+\cdots+\frac{1}{2^n}\tan\left(\frac{x}{2^n}\right) = \frac{1}{2^n}\cot\frac{x}{2^n}-\cot x. \tag{8}$$

- Binomial Theorem.
  - It is clear that

$$(a+b)^n = c_0\,a^0\,b^n + c_1\,a^1\,b^{n-1}+\cdots+c_n\,a^n\,b^0. \tag{9}$$

- $c_k =$ Number of ways to choose $k$ $a$'s from the $n$ $a's$. For example, $n=3, k=2$,

$$(a+b)\,(a+b)\,(a+b) \Longrightarrow a^2\, b; \tag{10}$$

$$(a+b)\,(a+b)(a+b) \Longrightarrow a^2\, b; \tag{11}$$

$$(a+b)(a+b)\,(a+b) \Longrightarrow a^2\, b. \tag{12}$$

  Therefore at the end we have $3\, a^2\, b$ in the expansion.

- Through counting we have

$$c_k = \binom{n}{k} := \frac{n!}{k!\,(n-k)!}. \tag{13}$$

  If we check the $c_0$ term in the expansion, we see that it makes sense to define $\binom{n}{0} = 1$.

- Therefore we have the binomial expansion

$$(a+b)^n = \binom{n}{0}\, a^0\, b^n + \cdots + \binom{n}{k}\, a^k\, b^{n-k} + \cdots + \binom{n}{n}\, a^n\, b^0 = \sum_{k=0}^{\infty} \binom{n}{k}\, a^k\, b^{n-k}. \tag{14}$$

**Problem 2.** Let $h > 0$ and $f: \mathbb{R} \mapsto \mathbb{R}$ be a function. One could define the "finite difference operator" as

$$(\triangle_h f)(x) := f(x) - f(x-h). \tag{15}$$

Further define

$$\triangle_h^2 f := \triangle_h (\triangle_h f), \qquad \triangle_h^3 f := \triangle_h(\triangle_h^2 f), \qquad \text{and so on.} \tag{16}$$

Prove

$$(\triangle_h^n f)(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} f(x - j\, h). \tag{17}$$

- Properties.
  - $\binom{n}{k} = \binom{n}{n-k}$. Note that this is consistent with the definition $\binom{n}{0} = 1$.
  - $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$.

  **Proof.** We have

$$\begin{aligned} \binom{n}{k} + \binom{n}{k-1} &= \frac{n!}{k!\,(n-k)!} + \frac{n!}{(k-1)!\,(n-k+1)!} \\ &= \frac{(n+1-k)\, n!}{k!\,(n-k+1)!} + \frac{k \cdot n!}{k!\,(n-k+1)!} \\ &= \frac{[(n+1-k)+k]\, n!}{k!\,(n-k+1)!} \\ &= \frac{(n+1)!}{k!\,(n+1-k)!} = \binom{n+1}{k}. \end{aligned} \tag{18}$$

  The proof ends. □

**Exercise 3.** Prove the binomial expansion theorem through induction with the help of the above identity.

- We prove

$$1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} > \left(1 + \frac{1}{n}\right)^n \tag{19}$$

  for every $n \in \mathbb{N}$.

**Proof.** Using binomial expansion we have

$$
\begin{aligned}
\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k 1^{n-k} \\
&= \sum_{k=0}^{n} \frac{n!}{k!\,(n-k)!\,n^k} \\
&= \sum_{k=0}^{n} \frac{n\,(n-1)\cdots(n-k+1)}{n^k}\,\frac{1}{k!} \\
&= \sum_{k=0}^{n} \left[\frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n}\right]\frac{1}{k!} \\
&< \sum_{k=0}^{n} [1\cdot 1\cdots 1]\,\frac{1}{k!} = 1+1+\frac{1}{2!}+\cdots+\frac{1}{n!}.
\end{aligned}
$$

Thus ends the proof. □