# MATH 117 FALL 2014 LECTURE 8 (SEPT. 15, 2014)

**Reading:** 314 Notes: Sets and Functions §1; Bowman §1.A.

- Sets.

  DEFINITION 1. *A set is a collection of objects. Each object is called a "member" (or "element") of this set.*

  NOTATION. *We use $a \in A$ to mean $a$ is a member of $A$ and use $a \notin A$ to mean $a$ is not a member of $A$.*

  - Empty set. There is exactly one set with no members at all.[1] We denote it by $\varnothing$.
  - Russell's paradox. Consider the set

    $$S := \{A \mid A \notin A\}. \tag{1}$$

    Now consider the question: Does $S \in S$?

    **Exercise 1.** Define a set $A$ such that $A \in A$. (Answer:[2] )

    **Problem 1.** Critique the following proof of the claim: There are only finitely many natural numbers.

    **Proof.** Consider the set

    $$A := \{\text{natural numbers that can be defined using less than } 10^6 \text{ English letters}\} \tag{2}$$

    $A$ is not empty as "the first natural number", which defines 1, is its member. Note that since there are only 26 English letters, there are only $26^{(10^6)}$ possible combinations of $10^6$ English letters and therefore there are only finitely many numbers in $A$. We now prove $A = \mathbb{N}$. More specifically, we prove the set

    $$B := \{\text{natural numbers that are not in } A\} \tag{3}$$

    is empty. We do this through proof by contradiction.

    Assume $B$ is not empty. Then among the numbers in $B$ there is a smallest one. But this number can be defined as "the smallest number that cannot be defined using less than one million English letters". Clearly this means this smallest number in $B$ should also be a member of $A$. This contradicts the definition of $B$. Therefore $B$ is empty and $A = \mathbb{N}$ and consequently $\mathbb{N}$ has no more than $26^{(10^6)}$ numbers. □

- Relations between sets

  - Subset. A set $A$ is a subset of another set $B$, denoted $A \subseteq B$, if and only if every member of $A$ is also a member of $B$.

    – Template for proving $A \subseteq B$:

      Take an arbitrary $a \in A$, [your argument here], we see that $a \in B$. Thus by definition $A \subseteq B$.

    **Example 2.** Let $A = \{n\,(n+1)\,(n+2) \mid n \in \mathbb{N}\}$ and $B = \{n \in \mathbb{N} \mid 3 \text{ divides } n\}$. Prove that $A \subseteq B$.

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1. Whether this needs proof depends on whether you take it as an axiom.

2. For example $A := \{\text{All the sets that can be defined with less than 100 English words}\}$. Then $A =$ "The set of all sets that can be defined with less than one hundred English words" which is 16 words, therefore $A \in A$.

**Proof.** Take an arbitrary $a \in A$. By definition of $A$, there is $n \in \mathbb{N}$ such that $a = n(n+1)(n+2)$. Now there are three cases for the remainder of $n \div 3$:

1. The remainder is 0. Then $3|n$ and therefore $3|[n(n+1)(n+2)]$ so $3|a$.
2. The remainder is 1. Then $3|(n+2)$ and we still have $3|a$.
3. The remainder is 2. Then $3|(n+1)$ and we still have $3|a$.

Thus in all situations we always have $3|a$ which means $a \in B$ by definition of $B$. Therefore $A \subseteq B$. □

- Do not confuse $\in$ and $\subseteq$. For example, if $A = \{1, 2, 3\}$, then $1 \in A$ but $\{1\} \notin A$, although it is true that $\{1\} \subseteq A$.
- In particular, $\varnothing \subseteq A$ for every set $A$.
- We also have $A \subseteq A$ for every set $A$.

  **Proof.** Take an arbitrary $a \in A$. Then $a \in A$ and therefore $A \subseteq A$. □

- One important property is transitivity:

$$\text{Assume } A \subseteq B, B \subseteq C, \text{ then } A \subseteq C. \tag{4}$$

  **Proof.** Take an arbitrary $a \in A$.

  Since $A \subseteq B$, by definition we have $a \in B$. This together with the definition of $B \subseteq C$ gives $a \in C$.

  Therefore $A \subseteq C$ by definition. □

- Equal. $A = B$ if and only if the two sets have the same elements.
  - To prove: Prove
    1. $A \subseteq B$;
    2. $B \subseteq A$.
- Proper subset. $A \subset B$ (or $A \subsetneq B$) if and only if
  1. $A \subseteq B$;
  2. $A \neq B$.

  Proving $A \subset B$ involves two steps.
  1. Prove $A \subseteq B$;
  2. Prove there is at least one element $b \in B$ such that $b \notin A$.

  **Example 3.** Let $A = \{n(n+1)(n+2) | n \in \mathbb{N}\}$ and $B = \{n \in \mathbb{N} | 3 \text{ divides } n\}$. Prove that $A \subset B$.

  **Proof.** We do this through the two steps.
  1. $A \subseteq B$. This has already been done above.
  2. There is at least one element $b \in B$ such that $b \notin A$. Take $b = 3$. Since $3|3$ we see that $3 \in B$. On the other hand, for every $n \in \mathbb{N}$ we have

$$n(n+1)(n+2) \geqslant 1 \times 2 \times 3 = 6. \tag{5}$$

Therefore every element of $A$ is greater than or equal to 6. As $3 < 6$ we see that $3 \notin A$.

Thus the proof ends. □

- Operations on two sets.

  Let $A, B$ be sets. Then we can create new sets through the following operations.

  - Union.

    The union $A \cup B$ is defined as $\{x | x \in A \text{ or } x \in B\}$. Note that here if $x$ is a member of both $A$ and $B$ then it is also a member of $A \cup B$.

    For example

    $$\{1,2,3\} \cup \{3,4,5\} = \{1,2,3,4,5\}. \tag{6}$$

  - Intersection.

    The intersection $A \cap B$ is defined as $\{x | x \in A \text{ and } x \in B\}$. For example

    $$\{1,2,3\} \cap \{3,4,5\} = \{3\}. \tag{7}$$

  - Difference.

    The difference $A - B$ (also can be denoted as $A \backslash B$) is defined as $\{x | x \in A \text{ but } x \notin B\}$. For example

    $$\{1,2,3\} - \{3,4,5\} = \{1,2\}; \qquad \{3,4,5\} - \{1,2,3\} = \{4,5\}. \tag{8}$$

    **Exercise 2.** Let $A, B, C, D$ be sets with $A \subseteq B, C \subseteq D$. Prove $A - D \subseteq B - C$.

    **Exercise 3.** Prove that

    $$A - B = A - A \cap B \tag{9}$$

    for any two sets $A, B$.

**Example 4.** Represent the set

$$A \triangle B := \{x | x \in A \text{ or } x \in B \text{ but not both}\}. \tag{10}$$

We see that

$$A \triangle B = \{x | x \in A \text{ or } x \in B\} - \{x | x \in \text{both } A, B\} = A \cup B - A \cap B. \tag{11}$$

We can also write

$$A \triangle B = \{x | x \in A \text{ but not } B\} \cup \{x | x \in B \text{ but not } A\} = (A - B) \cup (B - A). \tag{12}$$

**Exercise 4.** Prove directly

$$A \cup B - A \cap B = (A - B) \cup (B - A). \tag{13}$$

(Hint:[3] )

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3. We first prove that $A \cup B - A \cap B \subseteq (A - B) \cup (B - A)$. Take any $x \in A \cup B - A \cap B$. Then we have $x \in A \cup B$. Now there are two cases:

1. $x \in A$. Then we have $x \in A - A \cap B = A - B \subseteq (A - B) \cup (B - A)$. Note the equality is (9).
2. $x \in B$. Then we have $x \in B - A \cap B$ and still conclude $x \in (A - B) \cup (B - A)$.

Therefore we have $A \cup B - A \cap B \subseteq (A - B) \cup (B - A)$.