Math 117 Fall 2014 Lecture 7 (Sept. 12, 2014)

Reading: Bowman §1.H, §1.I, §1.J.

- A bit more about numbers.
  - $\mathbb{N}$: natural numbers; $\mathbb{Z}$: integers; $\mathbb{Q}$: rational numbers; $\mathbb{R}$: real numbers.
  - $\mathbb{Q}^c$: irrational numbers. More about the superscript c (for complement) next week.
  - $\mathbb{Q}$ is dense in $\mathbb{R}$; $\mathbb{Q}^c$ is dense in $\mathbb{R}$.

    Meaning: between any two real numbers there are at least one rational number and one irrational number.

    Exercise 1. Prove that between any two real numbers there are infinitely many rational numbers and infinitely many irrational numbers.

  - $\mathbb{Q}$ can be listed/counted as "the first rational, the second rational, ..." We know that there are infinitely many rational numbers and infinitely many natural numbers. It turns out that these two "infinities" are of the same size. On the other hand, real numbers cannot be listed/counted and the infinity for real numbers is much larger that the infinity for rationals/natural numbers.

    Exercise 2. Prove that irrational numbers cannot be listed/counted.

  - Some fun facts.

    Although we know $e, \pi \in \mathbb{Q}^c$ (we will prove $e \in \mathbb{Q}^c$ in 117 and maybe $\pi \in \mathbb{Q}^c$ in 118), it is not known whether $\pi + e, \pi e, \pi / e, e^e$ are rational or not.

  - A fun proof.

    Theorem 1. *There are $a, b \in \mathbb{Q}^c$ such that $a^b \in \mathbb{Q}$.*

    Proof. Consider $c := \sqrt{2}^{\sqrt{2}}$. There are two cases:

    1. $c \in \mathbb{Q}$. Then taking $a = b = \sqrt{2}$ we finish the proof;
    2. $c \in \mathbb{Q}^c$. Then taking $a = c, b = \sqrt{2}$ we have $a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = 2 \in \mathbb{Q}$ and still finish the proof. $\square$

  - A fun number. $e^{\pi\sqrt{163}}$ is almost an integer (262537412640768743.99999999999925...). See http://math.stackexchange.com/questions/4544/why-is-e-pi-sqrt163-almost-an-integer for explanation.
  - A fun theorem.

    Theorem 2. (N. Hegyvari 1993) *Let $n_1 < n_2 < n_3 < \cdots$ be natural numbers with $\sum_{k=1}^{\infty} \frac{1}{n_k} = \infty$. Then $0.n_1 n_2 n_3 \ldots \in \mathbb{Q}^c$.*

    Problem 1. Prove that 0.123456789101112131415... is irrational. (Hint: [1])

- Sets of numbers.

  Definition 3. *A set of numbers is a collection of numbers.*

1. To prove $1 + \frac{1}{2} + \frac{1}{3} + \cdots = \infty$, group $\frac{1}{3}$ and $\frac{1}{4}$ together, then $\frac{1}{5}, \ldots, \frac{1}{8}$ together, then $\frac{1}{9}, \ldots, \frac{1}{16}$ together and so on. This proof was discovered by the Middle Age philosopher Nicole Oresme (c. 1320 - 1325 – Jul. 11, 1382).

NOTATION. *Unless otherwise specified, by "numbers" we mean real numbers from now on.*

There are two ways to write a set.

1. List all its members. For example
$$A = \{1, 2, 3, 4\}. \tag{1}$$
Note that the order of this listing does not matter. Thus $\{1, 2, 3, 4\}$ and $\{4, 2, 3, 1\}$ are the same set.

   To check whether a number is a member of A, we just check whether it is in the list.

2. Specify the defining property. For example
$$A = \{x \mid x \in \mathbb{R}, x > 5\} \tag{2}$$
The properties that are satisfied by members of A and members of A only are $x \in \mathbb{R}$ and $x > 5$. Thus we have
$$1 + i \notin A, \qquad 3 \notin A, \qquad 5 \notin A, \qquad 7 \in A. \tag{3}$$

   NOTATION 4. *It is also OK to write A as $\{x \in \mathbb{R} \mid x > 5\}$. Or in many cases, when it's well understood that all the numbers under consideration are real numbers, we can just write $A = \{x \mid x > 5\}$.*

Exercise 3. Write down 5 numbers from each of the following sets.
$$A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}, \quad B = \left\{1 - \frac{1}{n} \mid n \in \mathbb{N}\right\}, \quad C = \{m^2 + n^2 \mid m, n \in \mathbb{N}\}. \tag{4}$$

- Upper bound and lower bound.

DEFINITION 5. *Let A be a set of numbers. We say $M \in \mathbb{R}$ is a upper bound of A, if and only if $M \geqslant a$ for every $a \in A$. We say $m \in \mathbb{R}$ is a lower bound of A, if and only if $m \leqslant a$ for every $a \in A$.*

Example 6. Let $A = \left\{\frac{1}{n} \mid n \in \mathbb{N}\right\}$.[2] Prove that 2 is an upper bound of A.

Proof. By definition of upper bound, it suffices to prove that for every $a \in A$, we have $a \leqslant 2$. Let $a \in A$ be arbitrary. By definition of A, there is $n \in \mathbb{N}$ such that $a = \frac{1}{n}$. Now as $n \geqslant 1$, we have
$$a = \frac{1}{n} \leqslant \frac{1}{1} \leqslant 2. \tag{5}$$
Thus ends the proof. $\square$

Important. It is a good idea to start a proof by citing relevant definitions. This way you will know exactly what you need to do. Many proofs in a beginning calculus class are in fact straightforward. But they may appear confusing if you do not form the habit of "starting from definitions".

Exercise 4. Prove that 0 is a lower bound for the set A in the above example.

2. This is a compact way of writing $A = \left\{x \mid x = \frac{1}{n} \text{ for some } n \in \mathbb{N}\right\}$.

Exercise 5. Let A be a set of numbers and $M \in \mathbb{R}$ be a upper bound of A. Prove: Any $M' > M$ is also a upper bound of A. Can you form and prove a similar claim for lower bounds?

Problem 2. Let $A = \left\{\sum_{k=0}^{n} \frac{1}{k!} | n \in \mathbb{N}\right\}$. Prove that 3 is a upper bound of A. (Hint:[3])

- Supreme and Infimum.

DEFINITION 7. *Let A be a set of numbers. Then the supreme of A, denoted* $\sup A$, *is defined as the smallest upper bound (least upper bound) of A, and the infimum of A, denoted* $\inf A$, *is defined as the largest (greatest) lower bound of A.*

  - The existence of the least upper bound of A is not obvious at all. In fact it is an assumption. We simply assume that $\mathbb{R}$ has the following "l.u.b. property": Let A be any set of real numbers. If it has a upper bound, then it has a least upper bound.
  - How to prove. To prove that a number M is the least upper bound of a set A, that is $M = \sup A$, we need to prove two things:
    1. M is a upper bound of A. That is for every $a \in A$, there holds $a \leqslant M$.
    2. M is the least among upper bounds. This is usually proved as follows: Any $M' < M$ is not a upper bound of A. To prove this, all we need to do is to find a particular number $a' \in A$ such that $a' > M'$.

    Exercise 6. Is it enough to find $a' \in A$ such that $a' \geqslant M'$?

    Exercise 7. Write down the two steps to prove a number m is the greatest lower bound of a set A.

  Example 8. Let $A = \left\{\frac{1}{n} | n \in \mathbb{N}\right\}$. Prove that $\sup A = 1, \inf A = 0$.

  Proof.

  - $\sup A = 1$.
    1. 1 is a upper bound of A.

       Exercise 8. Prove that 1 is a upper bound of A.

    2. 1 is the least among upper bounds. Let $b < 1$ be arbitrary. Now take the number from A to be $1 = \frac{1}{1} \in A$. We have $1 > b$ and therefore b is not a upper bound of A.
  - $\inf A = 0$.
    1. 0 is a lower bound of A.

       Exercise 9. Prove that 0 is a lower bound of A.

    2. 0 is the greatest lower bound of A. Let $b > 0$ be arbitrary. Take $n \in \mathbb{N}$ such that $n > b^{-1}$. Then we have $\frac{1}{n} \in A$ but $\frac{1}{n} < b$. So b is not a lower bound. □

3. $\frac{1}{n!} \leqslant \frac{1}{(n-1)\, n}$.