Math 117 Fall 2014 Lecture 6 (Sept. 11, 2014)

- What is π?
  - The ratio between the circumference and diameter of a circle; or the ratio between the area and the square of the radius of a circle. But why are they the same number?
  - The usual high school "proof" relies on two assumptions: As the number of sides increases,
    - The area of the polygon approaches that of the circle;
    - The circumference of the polygon approaches that of the circle.

    Both are subtle questions to answer. Will be proved in 217 or 317.
- Calculation of π through iteration schemes.
  - Nicolas of Cusa (1401 - 1464)

    Set $r_1 = 1, R_1 = \sqrt{2}$. Iterate

    $$r_{n+1} = \frac{r_n + R_n}{2}, \qquad R_{n+1} = \sqrt{R_n \cdot r_{n+1}}. \tag{1}$$

    Then r_n, R_n converges to the same limit r and $\pi = \frac{4}{r}$.

    Proof. We cannot really prove $\pi = \frac{4}{r}$ here but we could almost[1] prove that $r_n, R_n \longrightarrow r$ in two steps.

    - Step 1. r_n, R_n converges.

      We prove that r_n is increasing with upper bound and R_n is decreasing with lower bound. The convergence is then guaranteed by the least upper bound property of $\mathbb{R}$.

      We prove by induction the following claim:

      $$r_1 < r_2 < \cdots < r_n < R_n < \cdots < R_1 \tag{2}$$

      for every n. Once this is done we see that r_n is increasing with upper bound R_1, R_n is decreasing with lower bound r_1.

      - $n = 1$. Since $r_1 = 1 < \sqrt{2} = R_1$ the claim holds.
      - From n to $n + 1$. Assume

        $$r_1 < r_2 < \cdots < r_n < R_n < \cdots < R_1 \tag{3}$$

        we will prove

        $$r_1 < r_2 < \cdots < r_n < r_{n+1} < R_{n+1} < R_n < \cdots < R_1 \tag{4}$$

        It suffices to show $r_n < r_{n+1} < R_{n+1} < R_n$.

        We have

        $$r_{n+1} = \frac{r_n + R_n}{2} > \frac{r_n + r_n}{2} = r_n. \tag{5}$$

        $$r_{n+1} = \frac{r_n + R_n}{2} < \frac{R_n + R_n}{2} = R_n \tag{6}$$

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1. We do not have a rigorous definition of "converge" yet.

Applying (6) to $R_{n+1} = \sqrt{R_n \cdot r_{n+1}}$ we have

$$R_{n+1} = \sqrt{R_n \cdot r_{n+1}} < \sqrt{R_n \cdot R_n} = R_n \tag{7}$$

and

$$R_{n+1} = \sqrt{R_n \cdot r_{n+1}} > \sqrt{r_{n+1} \cdot r_{n+1}} = r_{n+1}. \tag{8}$$

Thus we have proved $r_n < r_{n+1} < R_{n+1} < R_n$.

- Step 2. The limits are the same.

  Denote by r, R the limits of r_n, R_n. Now take $n \to \infty$ in $r_{n+1} = \frac{r_n + R_n}{2}$. We have $r = \frac{r + R}{2}$ which immediately gives $r = R$.

The proof now ends. □

- Richard Brent and Eugene Salamin in 1975 (independently).

  Set $a_0 = 1, b_0 = \frac{1}{\sqrt{2}}, s_0 = \frac{1}{2}$ and now iterate:

$$a_k = \frac{a_{k-1} + b_{k-1}}{2}, \quad b_k = \sqrt{a_{k-1}\, b_{k-1}}, \quad c_k = a_k^2 - b_k^2, \quad s_k = s_{k-1} - 2^k\, c_k \tag{9}$$

and finally set

$$\pi_k = \frac{2\, a_k^2}{s_k}. \tag{10}$$

Then each iteration roughly doubles the correct digits in π_k.

- Calculation of π through infinite sereis.
  - John Wallis:

$$\frac{2}{\pi} = \frac{2 \times 2}{1 \times 3} \cdot \frac{4 \times 4}{3 \times 5} \cdot \frac{6 \times 6}{5 \times 7} \cdots \tag{11}$$

  - James Gregory:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots \tag{12}$$

  - Srinivasan Ramanujan:

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{k=0}^{\infty} \frac{(4\,k)!\,(1103 + 26390\,k)}{(k!)^4\, 396^{4k}} \tag{13}$$

  Each term gives 8 more correct digits.

  - David and Gregory Chudnovsky[2]:

$$\frac{1}{\pi} = 12 \sum_{k=0}^{\infty} (-1)^k \frac{(6\,k)!}{(3\,k)!\,(k!)^3} \frac{13591409 + 545140134\,k}{640320^{3k+3/2}}. \tag{14}$$

  Each term gives 14 more correct digits.

- BBP formula.

  In 1996, David H. Bailey, Peter Borwein and Simon Plouffe discovered the formula

$$\pi = \sum_{k=0}^{\infty} \frac{1}{16^k} \left(\frac{4}{8\,k+1} - \frac{2}{8\,k+4} - \frac{1}{8\,k+5} - \frac{1}{8\,k+6} \right) \tag{15}$$

2. See *The Mountains of Pi*, The New Yorker, Mar. 2, 1992 for the story of Chudnovsky brothers.

and realized that it allows computation of digits of π starting from any location without calculating the digits before this location. The catch is that the expansion of π here must be in base 16.

Exercise 1. Calculate the binary expansion of $11/3$ to four digits.

Example 1. A similar formula is the following, for $\ln 2$:

$$\ln 2 = \sum_{k=1}^{\infty} \frac{1}{k\,2^k}. \tag{16}$$

Assume that $\ln 2 = 0.a_1 a_2 a_3 \cdots$ in binary expansion. Let's say we would like to calculate a_3. Now what $\ln 2 = 0.a_1 a_2 a_3 \cdots$ means is that

$$\ln 2 = \frac{a_1}{2} + \frac{a_2}{4} + \frac{a_3}{8} + \cdots \tag{17}$$

Thus we see that

$$2^2 \ln 2 = (2\,a_1 + a_2) + \frac{a_3}{2} + \cdots \tag{18}$$

and $a_3 = 1$ if and only if the non-integer part of $2^2 \ln 2$ is no less than $1/2$ and $a_3 = 0$ if it is less than $1/2$. Now we have

$$2^2 \ln 2 = 2 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \cdots = 2 + \frac{1}{2} + \sum_{k=1}^{\infty} \frac{1}{(k+2)\,2^k}. \tag{19}$$

We notice that

$$0 < \sum_{k=1}^{\infty} \frac{1}{(k+2)\,2^k} < \sum_{k=1}^{\infty} \frac{1}{3\,2^k} = \frac{1}{3}. \tag{20}$$

Thus we know that $a_3 = 1$.

Problem 1. Can base 10 digits be calculated using these formulas?

- For more on π, check out
  - The World of Pi: http://www.pi314.net;
  - *π: A Biography of the World's Most Mysterious Number*, Alfred S. Posamentier, Ingmar Lehmann, Herbert A. Hauptman, Prometheus Books, 2004.

Exercise 2. Try to obtain (16) using Taylor expansion of $\ln(1+x)$.