Math 117 Fall 2014 Lecture 5 (Sept. 10, 2014)

- Prehistory.

  Before the invention of logarithm, people calculated multiplications through the following trigonometric identities:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B; \quad (1)$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B. \quad (2)$$

  From these one easily obtain

$$\sin A \sin B = \frac{1}{2} [\cos (A - B) - \cos (A + B)]. \quad (3)$$

  More specifically, say they would like to calculate $a \times b$, where a, b are between 0 and 1. They followed the procedure[1]

  1. Find A, B such that $\sin A = a$, $\sin B = b$.
  2. Calculate $A + B, A - B$.
  3. Find $\cos(A - B), \cos (A + B)$ and the result $a \times b$ follows immediately from (3).

  Note. These trig identities are important. Please remember them.[2]

- Exponent.

  Michael Stiffel noticed $2^m \cdot 2^n = 2^{m+n}$. Thus to calculate the product of $a = 2^m$ and $b = 2^n$, it suffices to find their "exponents" (representatives) m, n, add them up, and then find the number having "exponent" $m + n$. Such property had been known for a long time before Stiffel.

- John Napier (1550 - 1617).

  Napier defined logarithms physically. Imagine two particles moving along two parallel straight lines. One with speed proportional to its distance to the origin, and the other with constant speed. Then we can make the second particle an "exponent" of the first and call its location the "log" of the location of the first one at the same time.[3]

- Logarithm tables.

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1. This procedure is only practical because the existence of very accurate sin and cos tables. The reason why ancients wanted to calculate sine and cosine is that in their world view the sun, the moon and all the stars move along circles. Ideally the earth would be at the center. But this contradicted observations such as seasons are of different lengths. Therefore it may be necessary to move earth off the center. But where? To determine where earth should be it became necessary to calculate the length of the straight line segment connecting two points on the circle based on the angle it's facing. This is essentially sine. See *Was Calculus Invented in India* by David Bressoud in *The College Mathematics Journal*, Volume 33, No. 1, Jan. 2002, pp. 2 – 13.

2. If you are comfortable with complex numbers, then there is one simple way to remember. Recall that $e^{iA} = \cos A + i \sin A$, $e^{iB} = \cos B + i \sin B$. Then we have for example

$$\cos(A + B) + i \sin(A + B) = e^{i(A+B)} = e^{iA} e^{iB} = (\cos A + i \sin A) (\cos B + i \sin B) \quad (4)$$

   and the identities follow from the fact that $i^2 = -1$.

3. If we assume the speed of the second particle to be 1 and the speed of the first particle equals exactly the distance, then we would have (let y be the location of the first particle and x be that of the second)

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y \quad (5)$$

   which gives $y(x) = e^x$.

Exercise 1. What if the speed of the second particle is not 1?

The physical definition of Napier is not practical. To be able to calculate logarithms, tables linking numbers with their "exponents" must be built. We can re-write Stiffel's observation into a table:

$$\begin{array}{cccccccc} 1 & 2 & 4 & 8 & 16 & 32 & 64 & \cdots \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{array}$$

But this table is not practical in that it is too sparse: Most numbers do not appear in the first row – for example it cannot help us calculate 5×7.

The idea to fix this is to use powers of numbers closer to 1. For example, if we use 1.1, we would have

$$\begin{array}{cccccccccccccc} 1 & 1.1 & 1.2 & 1.3 & 1.5 & 1.6 & 1.8 & \cdots & 2.6 & \cdots & 9.0 & 9.8 & 10.8 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots & 10 & \cdots & 23 & 24 & 25 \end{array}$$

Note that it is not necessary to go beyond 10 as we can always pre-process the numbers a, b such that they are between 1 and 10.

This new table is definitely much more practical, but still is a bit too sparse. Thus we could try to use $1.01, 1.001,$ and so on. At the end of the day, Joost Burg built a table using $1+10^{-4}$ and Napier himself built one using $1+10^{-7}$.

- Through building such tables, it must be observed that

$$1.1^{10} \approx 1.01^{100} \approx 1.001^{1000}, \text{etc.} \tag{6}$$

Thus it is natural to suspect that there is a "limit" to the sequence of numbers $(1+10^{-k})^{10^k}$ or more generally $(1+1/n)^n$ as $n=1,2,3,\ldots$. This number is denoted e, and natural logarithm was defined as the "exponent" for any number with respect to this number e.

- Why does e exist?
  - We can show
    - $\left(1+\frac{1}{n}\right)^n$ is increasing when n gets larger and larger.
    - There is a number M such that for all $n\in\mathbb{N}$. $\left(1+\frac{1}{n}\right)^n < M$.

    In other words we can show that the sequence

$$(1+1)^1, \left(1+\frac{1}{2}\right)^2, \left(1+\frac{1}{3}\right)^3, \ldots \tag{7}$$

    increases with an upperbound (a "ceiling"). Naturally we expect there to be a "lowest possible ceiling", or "least upper bound". And this least upper bound, to which the sequence $\left\{\left(1+\frac{1}{n}\right)^n\right\}$ forever approaches but never reaches, is our number e.
  - Surprisingly the last step of our argument, the existence of this "least upper bound", cannot be proved and has to be assumed. In fact, the rigorous definition of $\mathbb{R}$, the set of real numbers, is an extension of $\mathbb{Q}$ in which least upper bound always exists.
  - Proof of $\left(1+\frac{1}{n}\right)^n$ is increasing.

    Proof. Since all these numbers are positive, it suffices to prove

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} > 1 \tag{8}$$

    for all $n\in\mathbb{N}$.

Now we calculate

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} = \left(1+\frac{1}{n+1}\right)\left[\frac{\left(1+\frac{1}{n+1}\right)}{\left(1+\frac{1}{n}\right)}\right]^{n} \tag{9}$$

$$= \left(1+\frac{1}{n+1}\right)\left[\frac{\left(\frac{n+2}{n+1}\right)}{\left(\frac{n+1}{n}\right)}\right]^{n} \tag{10}$$

$$= \left(1+\frac{1}{n+1}\right)\left(\frac{(n+2)\,n}{(n+1)^2}\right)^{n} \tag{11}$$

$$= \left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{(n+1)^2}\right)^{n}. \tag{12}$$

We apply the following "Bernoulli's inequality":

Let $x>-1$, $n\in\mathbb{N}$. Then $(1+x)^n \geqslant 1+n\,x$.

Setting $x=-\frac{1}{(n+1)^2}$ in (12) we see that $x>-1$ so Bernoulli's inequality applies. We have

$$\frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} = \left(1+\frac{1}{n+1}\right)\left(1-\frac{1}{(n+1)^2}\right)^{n} \tag{13}$$

$$\geqslant \left(1+\frac{1}{n+1}\right)\left(1-\frac{n}{(n+1)^2}\right) \tag{14}$$

$$= 1+\frac{1}{n+1}-\frac{n}{(n+1)^2}-\frac{n}{(n+1)^3} \tag{15}$$

$$= 1+\frac{1}{(n+1)^3}>1. \tag{16}$$

Thus the proof ends. □

- Proof of Bernoulli's inequality.

Proof. We prove by induction.

- The case $n=1$. In this case we have

$$(1+x)^n = 1+x = 1+n\,x \tag{17}$$

so the inequality holds.

- The case "If the inequality is true for n, then it is true for $n+1$". Assume that

$$(1+x)^n \geqslant 1+n\,x. \tag{18}$$

Multiply both sides by $1+x$. Since $x>-1$ we have

$$(1+x)^{n+1} \geqslant (1+n\,x)\,(1+x). \tag{19}$$

As

$$(1+n\,x)\,(1+x) = 1+n\,x+x+n\,x^2 \geqslant 1+(n+1)\,x \tag{20}$$

we have $(1+x)^{n+1}\geqslant 1+(n+1)\,x$, that is the inequality still holds for $n+1$.

Thus ends the proof. □

- Proof of the existence of an upper bound.

  Exercise 2. Prove that $(1+\frac{1}{n})^{n+1}$ is decreasing.

  Exercise 3. Prove that $(1+\frac{1}{n})^n$ has an upper bound. (Hint:[4])

- Other formulas for e.
  - Using $\left(1+\frac{1}{n}\right)^n$ to calculate e is not very efficient.
  - We have

$$e=1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\frac{1}{(n+1)!}+\cdots \tag{21}$$

where ! is "factorial", that is $2!=2\times 1, 3!=3\times 2\times 1, 4!=4\times 3\times 2\times 1$ and so on.

Exercise 4. Calculate e through

$$e\approx 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!} \tag{22}$$

with $n=3,6,9$. For each n find $m\in\mathbb{N}$ such that $(1+\frac{1}{m})^m$ has approximately the same accuracy.

  - Euler discovered the following two formulas:

$$e=2+\cfrac{2}{2+\cfrac{3}{3+\cfrac{4}{4+\cfrac{5}{5+\cdots}}}} \tag{23}$$

and

$$e=2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cdots}}}}}} \tag{24}$$

(the pattern is $2,1,1,4,1,1,6,1,1,8,1,1,\ldots$)

4. $\left(1+\frac{1}{n}\right)^n<\left(1+\frac{1}{n}\right)^{n+1}$ for every $n\in\mathbb{N}$. Now use the fact that $\left(1+\frac{1}{n}\right)^{n+1}$ is decreasing.