MATH 117 FALL 2014 LECTURE 4 (SEPT. 8, 2014)

- Reading:
  - Required reading: Dr. Bowman's book §1.B.
  - Optional reading: §1.C.
- $\sqrt{2}$ is irrational, that is $\sqrt{2}$ is not rational.

  Notation. The symbolic way of writing this is $\sqrt{2} \notin \mathbb{Q}$. (Recall that $\in$ means belongs to)
  - The significance of this.
    - $\mathbb{N}$: Justified by the need of counting;
    - $\mathbb{Z}$, $\mathbb{Q}$: Justified by the need to solve $a + x = b$, $a\, x = b$, which are universal in everyday life.
    - Presumably, by considering more complicated equations we could further extend the number system. But why should we do this? Why do we need numbers to satisfy equations like $x^2 - 2 = 0$ or $x^2 + 1 = 0$?
    - For $x^2 + 1 = 0$ the reason is more subtle, but for $x^2 - 2 = 0$, it's because the length of the diagonal of the unit square must satisfy this equation. Thus if we admit that:
      - Right angle is possible;
      - The Pythagorean Theorem is true;
      - For any finite line segment, its length is given by a number;

      Then we have to admit that this equation has a solution x which is a number. Therefore if the solution cannot be rational, we must admit the existence of "irrational" numbers.
    - The proof of the irrationality of $\sqrt{2}$ actually caused the ancient Greeks to turn away from algebra and focus on geometry.
  - The proof.

    Proof (of irrationality of $\sqrt{2}$). We prove by contradiction. Assume that there is a rational number solving $x^2 - 2 = 0$, then by definition of rational numbers, there are $p, q \in \mathbb{Z}$, $q > 0$, $(p, q) = 1$ such that

    $$\left(\frac{p}{q}\right)^2 - 2 = 0. \tag{1}$$

    Multiply both sides by q^2 (Note that this is OK because our $q > 0$) and move terms around, we reach

    $$p^2 = 2\, q^2. \tag{2}$$

    Thus p^2 is even. Consequently p is even. Thus there is $k \in \mathbb{Z}$ such that $p = 2\, k$. Substitute this back into (2) we have

    $$(2\, k)^2 = 2\, q^2 \tag{3}$$

    which simplifies to

    $$q^2 = 2\, k^2. \tag{4}$$

This means q^2 is even and consequently q is even. Now that both p, q are even, we have 2 to be a common divisor of p, q. This contradicts $(p, q) = 1$ which says the greatest common divisor of p, q is 1.

Example 1. Prove that $\sqrt{3}$ is irrational.

Proof. The proof is almost identical to that for $\sqrt{2}$, just change 2 to 3: We have

$$\left(\frac{p}{q}\right)^2 - 3 = 0 \tag{5}$$

and then

$$p^2 = 3\, q^2 \tag{6}$$

which means $3 | p^2$. Recall the corollary of the Fundamental Theorem of Arithmetic:

> If p is a prime, $a, b \in \mathbb{Z}$, then the following holds: If $p | (a\, b)$ then either $p | a$ or $p | b$.

Thus we have $3 | p$ or $3 | p$ which of course is simply $3 | p$. Therefore there is $k \in \mathbb{Z}$ such that $p = 3\, k$. Substitute this back into (6) we have

$$q^2 = 3\, k^2 \tag{7}$$

which means $3 | q$. So 3 is a common divisor to p, q and we reach a contradiction. □

Exercise 1. Prove the irrationality of $\sqrt{5}, \sqrt{7}, \sqrt{11}$.

Exercise 2. Prove the irrationality of $\sqrt[3]{2}$. [1]

Exercise 3. Prove the irrationality of $\sqrt{35}$.

Problem 1. Let $n \in \mathbb{N}$. Figure out for which n $\sqrt{n}$ is rational and for which n it is not. Justify your claim. (Meaning: You need to prove it).

Problem 2. Mr. Ben Pineau in our class provided the following clever proof for $\sqrt{3}$.

> If there are $p, q \in \mathbb{Z}$, $q > 0$, such that $p^2 = 3\, q^2$. We see that p, q are both even or both odd. Since $(p, q) = 1$, they have to be both odd. Write $p = 2\,k - 1, q = 2\,l - 1$ and substitute into the equation, we have
>
> $$4\,k^2 - 4\,k + 1 = 12\,l^2 - 12\,l + 3 \tag{8}$$
>
> which can be re-arranged into
>
> $$2\,[k^2 - k - 3\,l^2 + 3\,l] = 1. \tag{9}$$
>
> Note that the left hand side is even while the right hand side is odd. Contradiction.

Check whether this method can answer the question in Problem 1.

- Calculation of $\sqrt{2}$.
  - Method 1 (Ancient Babylon).

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1. This is also a famous number, in that it not only is irrational, but also is not constructible, that is cannot be constructed through straightedge and compass (meaning, if you are given a line segment, you cannot construct another line segment whose length is $\sqrt[3]{2}$ times that of the original segment). Note that $\sqrt{2}$ is constructible – Try to show this! As the story goes, there was a plague in the city state of Delos. Somehow the people there believed that the cause was Apollo and the solution was to double the size of the cubic altar. Like every normal person, they doubled the sides of the altar but the plague did not stop!! They then realized that the correct side length should be $\sqrt[3]{2}$ but of course had not idea how to obtain such a length. Thus the famous "doubling the cube" problem. That $\sqrt[3]{2}$ is not constructible was proved by Pierre Wantzel in 1837. Unfortunately, the key ingredient of the proof is not calculus but Galois theory, part of abstract algebra.

Take any $a_1 > 0$ with $a_1^2 \neq 2$. Set $b_1 = 2/a_1$. Now iterate

$$a_{n+1} = \frac{1}{2}(a_n + b_n), \qquad b_{n+1} = \frac{1}{\frac{1}{2}\left(\frac{1}{a_n} + \frac{1}{b_n}\right)}. \tag{10}$$

Then $a_n, b_n \to \sqrt{2}$.

Exercise 4. Prove the following.

a) For every n, $a_n b_n = 2$;

b) For every $n \geqslant 2$, $a_n > b_n$;

c) When $n \geqslant 2$ a_n is decreasing and b_n is increasing. That is $a_2 > a_3 > a_4 > \cdots, b_2 < b_3 < b_4 < \cdots$.

Exercise 5. Modify the algorithm for the calculation of $\sqrt{7}$.

- Method 2 (Newton's method)

  Take any $x_1 > 0$. Iterate

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}. \tag{11}$$

Then $x_n \longrightarrow \sqrt{2}$.

Exercise 6. Critique on the following "proof":

> Let $x \in \mathbb{R}$ be the real number that is the limit of x_n. Then taking $n \to \infty$ in (11) we have
>
> $$x = \frac{x}{2} + \frac{1}{x} \tag{12}$$
>
> which gives $x^2 = 2$. Therefore the limit of x_n is $\sqrt{2}$.

- Method 3 (Continued fractions)

  We have

$$\sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{2 + \frac{1}{\cdots}}}}}. \tag{13}$$

Exercise 7. Calculate a few terms

$$1 + \frac{1}{2}, 1 + \frac{1}{2 + \frac{1}{2}}, \ldots. \tag{14}$$

to provide numerical evidence for (13).

Exercise 8. Suppose we can treat such infinite fractions as a usual number, find a simple way to see that the continued fraction should be $\sqrt{2}$. (Hint:[2])

Remark 2. It is important to realize that for infinite processes like the $\{x_n\}$ above or the right hand side of (13), it is not guaranteed that we could manipulate it as a usual number. Such manipulation may lead to results like

$$1 + 2 + 3 + 4 + \cdots = -\frac{1}{12} \tag{15}$$

which only started to make sense when interpreted in a certain way related to quantum mechanics. See e.g. this numberphile video: http://youtu.be/w-I6XTVZXww.

Although (15) got "saved" by quantum mechanics, in most situations we should be more careful and prove "convergence" – that is show that there is indeed a certain number that is related in an appropriate way to the infinite process – first.

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2. Set $x = 1 + \frac{1}{2 + \frac{1}{\cdots}}$ and find an equation for x. Note that this is **not** a proof of (13).