UNIVERSITY OF ALBERTA
MATHEMATICAL AND STATISTICAL SCIENCES
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Assignment 4, due March 15, 2012, in class

Exercise 15: (Diffusion through a membrane)[1] (2)

We assume that a membrane of width L separates two regions (e.g., the interior and exterior of a cell). Consider a chemical that has a concentration c_1 inside the cell and c_2 outside the cell. The transport through the membrane can be described by a one-dimensional diffusion equation $u_t = Du_{xx}$. We assume that the solution settles onto an equilibrium.

(a) Find the equilibrium and sketch the concentration at equilibrium as a function of position.

(b) Using Fick's law, the flux across the membrane is given as $J(x,t) = -Du_x(x,t)$. Find the flux at equilibrium. The quotient D/L is known as the permeability of the membrane. Why do you think this is so?

Exercise 16: (Signalling in Ant populations)[2] (6)

Certain ant species (such as *Pogonomyrmex badius*) use pheromones as a signal for danger. In experiments, Bossert and Wilson released ants in a long tube and they stimulated one ant until it released a pheromone. They measure within which distance and after which time delay the other ants would react to the signal. A good model for the spread of the pheromones in the tube is the one-dimensional diffusion equation. We assume that at time $t = 0$ a signal of strength α is released. The diffusion constant is $D = 1$. Other ants react to the stimulus if the concentration they perceive is 10% of α or higher.

(a) For each $t > 0$, find the region in the tube $0 \le x \le x(t)$ where the ants would react to the stimulus (region of influence).

(b) Sketch the time evolution of $x(t)$.

(c) Find the time t^* such that the region of influence is empty for all $t > t^*$.

Exercise 17: (Dingos in Australia)[3] (4)

A Dingo population which lives in the eastern parts of Australia is prevented from invasion to the west by a fence which runs north-south. In this exercise, we study the case that the fence breaks down somewhere (at time $t = 0$). Two farms, A and B, are located on the west side of the fence. The distance from farm A to the fence

[1] Exercise 4.5.1 from de Vries et al, 2006
[2] Exercise 4.5.3 from de Vries et al, 2006
[3] Exercise 4.5.4 from de Vries et al. 2006

is 100 miles and the distance from farm A to B is another 100 miles. The farmers would like to know how long it would take for the Dingos to reach their farms. We model the spread of the Dingo population with the Fisher equation

$$u_t = Du_{xx} + ku\left(1 - \frac{u}{K}\right)$$

with $k = 1$ (1/month), and $K = 1$ (in the units of u).

1. In the region between the fence and farm A the land is flat and the diffusion constant is $D_1 = 100$ (miles^2/month). Since at time $t = 0$ there are no Dingos west of the fence, the corresponding invasion wave will have the minimal wave velocity c^*. When does the Dingo population reach farm A?
2. The region between farm A and B has rocks and slope, hence there the diffusion constant is $D_2 = 50$ (miles^2/month). At farm A, the travelling wave has already assumed an exponential leading edge with decay rate λ. Now, as the diffusion constant changes, we assume the wave form stays the same (leading edge of $e^{-\lambda x}$), while the invasion speed changes. When does the Dingo population reach farm B?

Exercise 18: (Signal transport in an axon)[4] (8)

Fitzhugh and Nagumo derived a model for signal transduction in the axon,

$$u_t = u_{xx} + u(1-u)\left(u - \frac{1}{2}\right),$$

where $u(x,t)$ represents the membrane potential. We study this model on $0 \leq x \leq 1$ with homogeneous Neumann boundary conditions,

$$u_x(0,t) = 0 \qquad u_x(1,t) = 0.$$

(a) Determine the system of two ODEs which describe the steady state solutions.

(b) Find the equilibrium points of the system of (a) and study their stability.

(c) Show that

$$H(u, u_x) = \frac{1}{2}u_x^2 - \frac{1}{4}u^4 + \frac{1}{2}u^3 - \frac{1}{4}u^2$$

is a Hamiltonian function for the system you found in (a).

(d) Sketch a phase portrait in the (u, u_x)-plane.

(e) In the phase portrait, find the steady state solutions that satisfy the Neumann boundary conditions, and sketch them as a function of x.

(f) Give a biological interpretation of these steady state solutions.

[4] Exercise 4.5.5 from de Vries et al., 2006