University of Alberta
Mathematical and Statistical Sciences
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**Math 570, Mathematical Biology**
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**Assignment 2, due Feb 07, 2012, in class**

**Exercise 6:** (World population) (8)

The size of the world population (in million) between 1850 and 1997 is given in the following table

| Year | 1850 | 1940 | 1950 | 1960 | 1970 | 1985 | 1997 |
|---|---|---|---|---|---|---|---|
| Pop. | 1200 | 2249 | 2509 | 3010 | 3700 | 4800 | 5848.7 |

where we assume that the data points are independent, normally distributed and with constant variance $\sigma^2 = 100^2$. Three models are proposed to fit these data:

$$\begin{aligned} &\text{model 1:} && y = a_1 x + b_1 \\ &\text{model 2:} && y = c_2 x^2 + a_2 x + b_2 \\ &\text{model 3:} && y = d_3 x^3 + c_3 x^2 + a_3 x + b_3 \end{aligned}$$

1. Are these models nested? What method do you use to fit the model parameters?
2. Find the parameters of best fit to the data, and plot the best fit and the data in a common graph.
3. Compare the models using the AICc.
4. Which model would you choose? Would you be satisfied with the result? If not, why not?

**Exercise 7:** (Exponential sojourn function) (3)

Let $\mathcal{F}$ be a differentiable survival function that satisfies

$$\mathcal{F}(a+b) = \mathcal{F}(a)\mathcal{F}(b), \qquad \text{for all } a, b \geq 0.$$

Prove the following alternative. Either $\mathcal{F}(a) = 0$ for all $a \geq 0$, or there exists a $\mu > 0$ such that $\mathcal{F}(a) = e^{-\mu a}$.

**Exercise 8:** (Expected remaining sojourn) (3)

Assume that in an experiment you found that the *expected remaining survival time of an individual of age* $a$ is given by

$$D(a) = \kappa e^{-a}.$$

1. In class, we proved a Lemma, which gives a condition on $D(a)$ such that $D(a)$ can be used to define a sojourn function. Find a condition on $\kappa$ such that $D(a)$ qualifies to be an expected remaining sojourn function.
2. Compute the corresponding sojourn function $\mathcal{F}(a)$ and the death rate $\mu(a)$.

**Exercise 9 :** (Moivre survival function) (3)

Consider a generalized Moivre survival function

$$\mathcal{F}(a) = \left[1 - \frac{a}{c}\right]_+^\kappa$$

with finite maximum age $c > 0$. Show the following relation between the maximum age $c$, the life expectancy $D$ and the average expectation of remaining life $E$:

$$\frac{c}{D} = \frac{E}{D - E}.$$

**Exercise 10:** (Fixed stage duration) (3)

For a structured population model assume $\mathcal{F}(a) = 1$ for $0 \leq a < c$ and $\mathcal{F}(a) = 0$ for $a > c$. Let $B(t)$ and $u_0(t)$ be continuous. Show that $N(t)$ is differentiable for $t \neq c$ and

$$N'(t) = \begin{cases} B(t) - u_0(c - t), & 0 < t < c \\ B(t) - B(t - c), & t > c. \end{cases}$$