University of Alberta
Mathematical and Statistical Sciences
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Math 570, Mathematical Biology
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Assignment 1, due Jan 24. 2012, in class

Exercise 1: (5)

Submit a written (1-2 pages) proposal about a possible full term research project.

- What topic will you be working on?
- Where is this topic situated in the broad area of mathematical biology?
- Mention some relevant background and literature.
- What specific questions will you focus on in your investigation?
- What is the relevance/significance of these questions?
- What attracted you to the topic?

Exercise 2: (Self intoxicating population) (4)

Some populations (such as algae and bacteria) produce waste products, which in high concentrations are toxic to the population itself. A typical mathematical model for a population $n(t)$ and a toxic waste product $y(t)$ is

$$\begin{aligned} \dot{n} &= (\alpha - \beta - Ky)n \\ \dot{y} &= \gamma n - \delta y \end{aligned}$$

with $\alpha, \beta, \gamma, \delta, K > 0$.

1. Explain each term in the above model and sketch an arrow diagram.
2. Find the nullclines, the steady states and sketch a phase portrait and the vectorfield.
3. Linearize the system at the steady states and determine their stability. Find the regions in parameter space such that the nontrivial (coexistence) equilibrium is a node or a spiral.
4. Sketch some trajectories for the case of $\delta < 4(\alpha - \beta)$ and explain what you see in terms of biology.
5. Consider the case of high dilution $\delta >> 1, \gamma/\delta < \infty$.

Exercise 3: (Reaction kinetics) (2)

Write down differential equation models for the following pathways:

1. $$A \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} B \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} C \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} A$$

2. $$S + E \underset{k_{-1}}{\overset{k_1}{\rightleftharpoons}} C_1 \underset{k_{-2}}{\overset{k_2}{\rightleftharpoons}} C_2 \underset{k_{-3}}{\overset{k_3}{\rightleftharpoons}} E + P$$

Exercise 4: (Singular perturbation) (5)

Use the method of singular perturbation to understand the qualitative behavior of the following system:

$$\begin{aligned} \dot{x} &= x(1-y)(2-x), \\ \varepsilon\dot{y} &= x - y. \end{aligned}$$

Identify the fast and the slow system. Solve for the inner and outer solutions, find the slow manifold and check the matching conditions.

Exercise 5: (Maximum Likelihood) (4)

Consider data points $k_1, \ldots, k_n$ which are n independent realizations of a Poisson distributed random variable K, i.e.

$$P(K = k) = \frac{\lambda^k}{k!} e^{-\lambda}$$

1. Derive the likelihood for the data points k_i, the log-likelihood and the maximum likelihood estimator for λ.
2. What is the interpretation of λ?
3. Assume that you observed the infectious period of 8 infected individuals as: 3 days, 5 days, 4 days, 4 days, 5 days, 4 days, 4 days, 7 days. Assume that the infectious period is Poisson distributed and estimate the recovery rate λ.
4. Compute the log-likelihood for these data.
5. Show the data and the model in a common graph.
6. Is the assumption of Poisson distributed recovery appropriate? Would an exponential distribution be better?