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If we assume, by contradiction, that dim N ⩾ 1, the operator $\frac{\partial}{\partial t}$ has at least one non zero eigenvector : $u \in N$, $u \neq 0$ and $u' = \lambda u$. Consequently $u = e^{\lambda t}\psi$ where $\Delta\psi = \lambda\psi$. Since $\psi = \frac{\partial\psi}{\partial\nu} = 0$ on Γ_0, an open subset of Γ, one has $\psi \equiv 0$ on an open subset ω of Ω provided Γ is piecewise analytic. If Ω is connected we conclude that $u \equiv 0$, a contradiction.

Then if (7) were not true this would imply dimN ⩾ 1. From (1') and (7) we deduce (1) and exact controllability.

6. Conclusion

The conclusion is as follows. The four assumptions

(i) $D\mathbf{m} + D\mathbf{m}^* > 0$ on $\overline{\Omega}$

(ii) Ω is connected and Γ is piecewise analytic (or even C^2)

(iii) $\mathbf{m}.\boldsymbol{\nu} = 0$ at crack tips,

(iv) $\mathbf{m}.\boldsymbol{\tau} > 0$ at crack tips

imply exact controllability. If in addition $\mathbf{m}.\boldsymbol{\nu} \leqslant 0$ along the cracks, the control v vanishes on the cracks as one would reasonably expect.

The extra flexibility allowed by these possible choices of **m** yields more general distribution of the cracks than the very particular choice of **m** considered in the above §3. Various examples of such multipliers **m** are given in Triggiani (1988).

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Differential Equations on Branched Manifolds

K. P. HADELER and THOMAS HILLEN Tübingen University, Tübingen, Germany

Summary: Vector fields on two-dimensional branched manifolds can be seen as caricatures of three-dimensional problems. The corresponding semiflows are easy to visualize locally because of dimension two, on the other hand they are not continuous. With suitable transversality conditions one can obtain information on qualitative behavior and limit sets of trajectories. Particular attention is given to the fact that limit sets, in general, need not be invariant. Typical bifurcations are studied. The phenomena and applications are illustrated by graphical and numerical examples.

Introduction

The qualitative behavior of differential equations on two-dimensional compact manifolds is relatively well understood (see, e.g., Palis and di Melo [11]) whereas the behavior of differential equations in three dimensions can be very complicated and difficult to visualize. Therefore branched manifolds have been used as two-dimensional caricatures of three-dimensional systems. Williams [15] has developed a general concept of branched manifolds and he has studied caricatures of the Lorenz attractor ([16], [17], see also Guckenheimer and Holmes [5] and Sparrow [13]). One can make a similar construction for the Rössler attractor ([12], see also Jetschke [7]). Hadeler and Shonkwiler [6] have used branched manifolds in an epidemiological model. There is a rather close connection between differential equations on branched manifolds and differential equations with reset conditions or, in other terminology, differential equations with discontinuous right hand sides. Filippov [4] has developed a theory of such differential equations and has collected a vast bibliography. His view is essentially restricted to local qualitative analysis whereas we shall try to take a global view. In an abstract setting differential equations with discontinuous right hand sides can also be seen as differential inclusions (see, e.g., Aubin and Cellina [2]) or systems for set-valued functions.

Mostly a branched manifold has been seen as a set of planes (with boundaries) joined together along certain edges to form a geometric object on which vector fields are studied. At the edges transition conditions have been defined ad hoc ("if the trajectory ... "). In this view the connectedness and, in particular, the embedding in three dimensional space play a major role. Here we follow a more abstract approach which provides a rigorous definition of transition conditions. The appropriate construction requires some effort. Later we shall return to the familiar object by way of identifications.

Definition of a branched manifold

The purpose of branched manifolds is, of course, to study vector fields and their trajectories. In our approach (as in Williams [15]) we define a branched manifold as a geometric object independent of any vector fields or flows. Roughly speaking a branched manifold is a collection $\tilde{\mathcal{M}}$ of smooth manifolds M_i with transition conditions. Trajectories "run" on one manifold, "arrive" at some submanifold, and "continue" on another manifold.

If the transition conditions are used to identify points on the manifolds M_i one obtains an object $\mathcal{M}$ which heuristically can be seen as a set of surfaces glued together along lines. Then the transition conditions ensure that at each point of $\mathcal{M}$ there is a well-defined tangent space.

In this approach one can, on a fixed branched manifold, study vector fields depending on parameters and related bifurcations. In another view, used in [6], one can start from given vector fields on the M_i and study changes in the transition conditions.

In applications the constituting M_i will be mostly spheres S^2, and the N_{ij} will be spheres S^1.

In realistic applications the manifolds will be simply connected planar domains, and the submanifolds will be line segments. However, these manifolds are topologically equivalent to spheres. As usual in differential equations the assumption of compact manifolds without boundary merely leads to some simplification of the representation and some unification, it makes a coherent theory possible.

Let M_i, $i = 1, \ldots, r$, be two-dimensional compact C^1 manifolds without boundary.

Let N_{ij}, $j = 1, \ldots, s_i$, be one-dimensional (compact) C^1 submanifolds of M_i. Assume

$$N_{ij} \cap N_{ik} = \emptyset \quad \text{for} \quad j \neq k. \tag{1}$$

We define

$$M_i' = M_i \setminus \bigcup_{j=1}^{s_i} N_{ij}. \tag{2}$$

Now we define transition conditions. It is useful to introduce an index set $J = \{(i,j) : i = 1, \ldots, r;\ j = 1, \ldots, s_i\}$. One can visualize J as a matrix with rows of different lengths. Furthermore denote by L the set $L = \{1, \ldots, r\}$. Let $V : J \to L$ be a function.

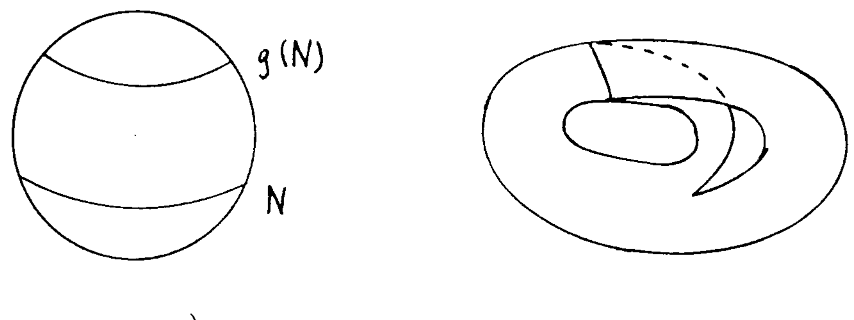
For $(i,j) \in J$ let g_{ij} be a C^1 mapping $g_{ij} : N_{ij} \to M_l$ where $l = V(i,j)$. By construction the image $g_{ij}(N_{ij}) \subset M_l$ is compact.

The main hypothesis on the various manifolds is the following nonintersection property. For $l \in L$ we require

$$\left(\bigcup_{V(i,j)=l} g_{ij}(N_{ij}) \right) \cap \left(\bigcup_{j=1}^{s_l} N_{lj} \right) = \emptyset. \tag{3}$$

The object $\tilde{\mathcal{M}} = \{M_i, N_{ij}, g_{ij} : (i,j) \in J\}$ is called a branched manifold. In addition to $\tilde{\mathcal{M}}$ we shall need the underlying point set $M = \cup_{i=1}^{r} M_i$.

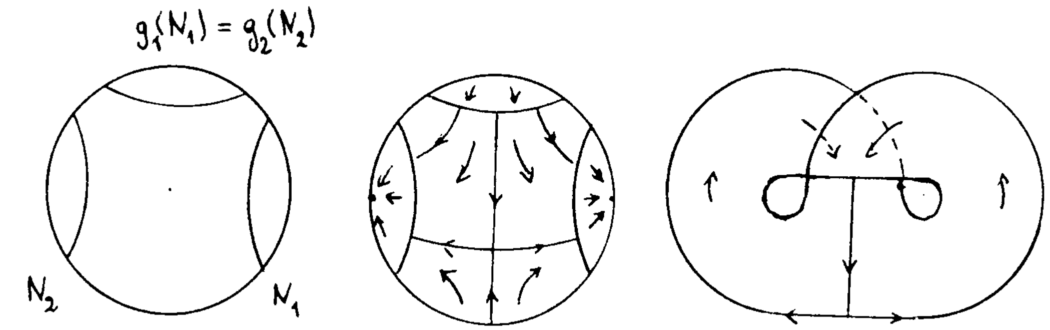
Example 1: $r = 1$, $s = 1$. Thus V maps $(i,j) = (1,1)$ into $l = 1$. There is only one manifold $M = M_1 = S^2$, and only one submanifold $N = N_{11}$, and $g : N \to M$. Condition (3) reduces to $g(N) \cap N = \emptyset$. This example corresponds to the classical reset problem. The trajectory runs on M until it meets N. Then it is reset to $g(N)$ and starts again (see Fig.1a). The caricature of the dynamics of the Rössler attractor as given by Jetschke [7] fits into this scheme (Fig.1b).



a) b)

Fig.1: The reset problem as a branched manifold

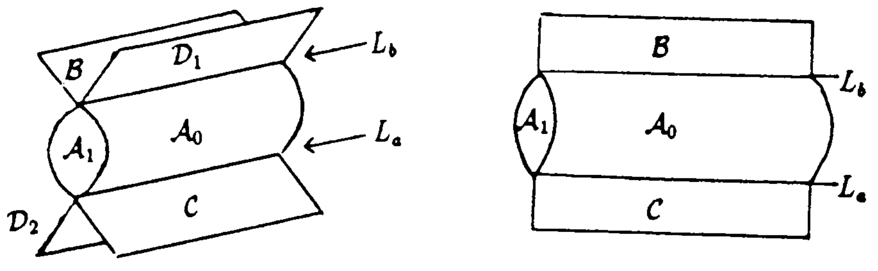
Example 2: $r = 1$, $s = 2$, $M_1 = S^2$, $V(1,1) = 1$, $V(1,2) = 1$. N_1, N_2 are disjoint sets S^1, and the images $g_1(N_1)$, $g_2(N_2)$ coincide (as sets). $g_1(N_1) = g_2(N_2)$ is a sphere S_1 disjoint from $N_1 \cup N_2$ (Fig.2a). The Fig.2b shows a flow on M. The shaded area in Fig.2b is equivalent to William's caricature of the Lorenz attractor (Fig.2c).



a) b) c)

Fig.2: The Lorenz attractor

Example 3: $r = 2$, $s_1 = s_2 = 1$, $M_1 = S^2$, $M_2 = S^2$, $V(1,1) = 2$, $V(2,1) = 1$. This setting is the common situation in problems of pest control or epidemic control ([6]) where there is a switch between strategies. Fig.2a presents the general situation for spheres. In the case studied in [6] one cap of each sphere can be discarded since no trajectory ever returns to these caps. Then the part of the branched manifold which contains the essential dynamics can be represented as in Fig.2b.



a) b)

Fig.3: Epidemic control problem as a branched manifold

On $\tilde{\mathcal{M}}$ (i.e., on M endowed with the structure of $\tilde{\mathcal{M}}$) we introduce an equivalence relation $\sim$ in the following way. Let $\mathcal{G}$ be the set of functions $\mathcal{G} = \{g_{ij} : (i,j) \in J\} \cup \{id\}$ where id is the identity on M. For points $\tilde{x}, \tilde{y} \in \tilde{\mathcal{M}}$ we define $\tilde{x} \sim \tilde{y}$ if there are functions $f, g \in \mathcal{G}$ such that $f(\tilde{x}) = g(\tilde{y})$. One can visualize the identification by $\sim$ as glueing together the M_i along N_{ij} and $g(N_{ij})$, respectively.

Proposition 1: *The relation $\sim$ is an equivalence relation.*

Proof: $\tilde{x} \sim \tilde{x}$ since $id \in \mathcal{G}$. $\tilde{x} \sim \tilde{y}$ implies $\tilde{y} \sim \tilde{x}$ by symmetry of definition. Assume $\tilde{x} \sim \tilde{y}$ and $\tilde{y} \sim \tilde{z}$. Then there are functions $f, g, h, k \in \mathcal{G}$ such that $f(\tilde{x}) = g(\tilde{y})$ and $h(\tilde{y}) = k(\tilde{z})$. In view of (1) there are only two cases.
Case 1: $g = h$. Then $f(\tilde{x}) = h(\tilde{z})$, $\tilde{x} \sim \tilde{z}$.
Case 2: $g = id$ (or $h = id$). Then $f(\tilde{x}) = \tilde{y}$ and $h(\tilde{y}) = k(\tilde{z})$. $h = id$ would lead to Case 1. Thus assume $h \neq id$. From (3) it follows that $f = id$, $\tilde{x} = \tilde{y}$, $h(\tilde{x}) = k(\tilde{z})$, $\tilde{x} \sim \tilde{z}$.

Now we define the object $\mathcal{M} = \tilde{\mathcal{M}}/\sim$. The set $\mathcal{M}$ will be endowed with the quotient topology and thus it becomes a compact Hausdorff space. We shall call both objects $\mathcal{M}$ and $\tilde{\mathcal{M}}$ the branched manifold. We shall denote by π the natural projection $\pi : \tilde{\mathcal{M}} \to \mathcal{M}$.

The object $\mathcal{M}$ looks rather similar to the ramified spaces which have been studied by Lumer [8], v.Below [3], Nicaise [9], Ali-Mehmeti [1] in connection with diffusion equations. We underline that they are indeed different. Ramified spaces are also constructed from smooth manifolds by way of identifications, but in their construction there is no directional information. Of course the present notion of branched manifold is closely related to the notion of Williams [15]. In Williams' definition of a branched manifold there is, by definition, a well-defined tangent space at every point. The natural visualization of Williams' branched manifold is a set of two-dimensional surfaces in $\mathbb{R}^3$ which pass through a curve such that at each point of this curve all surfaces have the same tangent plane. In the present context the appropriate visualization is given by Figs.1 and 2. Here the uniqueness of the tangent space is a consequence of the construction as it will be shown in the next proposition.

Proposition 2: *In each equivalence class $x \in \mathcal{M}$ there is one and only one point $\tilde{x}$ with the property:*

$$\left\{\tilde{y} \in \tilde{\mathcal{M}},\ f \in \mathcal{G},\ f(\tilde{x}) = \tilde{y}\right\} \Rightarrow f = id. \tag{4}$$

Proof: If x contains only one point $\tilde{x}$ then $\tilde{x}$ has property (4). Suppose x contains two points $\tilde{y}, \tilde{z} \in M$ with $\tilde{y} \neq \tilde{z}$. Then there are $f, g \in \mathcal{G}$ such that $g(\tilde{y}) = h(\tilde{z})$. Put $\tilde{x} = g(\tilde{y})$. Then $\tilde{x} \in x$ because $id \in \mathcal{G}$. The point $\tilde{x}$ has property (4) in view of condition (3). Now assume there are $\tilde{x}_1, \tilde{x}_2 \in x$ with property (4). Then there are functions $h_1, h_2 \in \mathcal{G}$ such that $h_1(\tilde{x}_1) = h_2(\tilde{x}_2)$. Then, again by (4), $h_1 = h_2 = id$ and $\tilde{x}_1 = \tilde{x}_2$.

Definition: For each equivalence class $x \in \mathcal{M}$ the point $\tilde{x} \in \mathcal{M}$ with property (4) is called the actual point $\alpha(x)$. The tangent space at $x \in \mathcal{M}$ is given by $T_x\mathcal{M} = T_{\alpha(x)}\tilde{\mathcal{M}}$.

Thus $\alpha : \mathcal{M} \to \tilde{\mathcal{M}}$ is the map which attributes to each point on $\mathcal{M}$ the actual point in $\tilde{\mathcal{M}}$. An equivalence class or point $x \in \mathcal{M}$ is called trivial if it contains only one point of $\tilde{\mathcal{M}}$. Otherwise the point is called a branch point. The set of trivial equivalence classes is open in $\mathcal{M}$. The function α is continuous on this open set. It should be underlined that π does not induce a natural projection of $T\tilde{\mathcal{M}}$ to $T\mathcal{M}$.

Flows

Suppose that on each manifold M_i, $i = 1, \ldots, r$, there is a C^1 vector field f_i. This collection of vector fields defines a C^1 vector field $\tilde{f}$ on M and $\tilde{\mathcal{M}}$. By integrating the vector fields f_i we obtain flows $\Phi_i(t, x)$, $i = 1, \ldots, r$ which exist for all $t \in \mathbb{R}$. The function $\Phi_i(t, x_0)$ is the solution of

$$\dot{x}(t) = f_i(x(t)), \quad x(0) = x_0, \tag{5}$$

i.e.,

$$\begin{aligned} \frac{\partial \Phi_i(t,x)}{\partial t} &= f_i(\Phi_i(t,x)), \\ \Phi_i(0,x) &= x. \end{aligned} \tag{6}$$

We want to construct a semiflow $\tilde{\Phi}$ on $\tilde{\mathcal{M}}$ which for small $t > 0$ has the property

$$\tilde{\Phi}(t, \tilde{x}) = \begin{cases} \Phi_i(t, \tilde{x}) & \text{if } \tilde{x} \in M_i', \\ \Phi_l(t, g_{ij}(\tilde{x})) & \text{if } \tilde{x} \in N_{ij},\ V(i,j) = l. \end{cases} \tag{7}$$

Theorem 3:
i) There is a unique semiflow $\tilde{\Phi}$ *on* $\tilde{\mathcal{M}}$ *which has the property* (7).
ii) The function

$$\Phi(t,x) = \pi\tilde{\Phi}(t,\alpha(x)) \tag{8}$$

defines a semiflow Φ *on* $\mathcal{M}$ *for which* $\tilde{\Phi}(t,\alpha(x)) = \alpha(\Phi(t,x))$ *has the property* (7).

Proof: Define, for $\tilde{x} \in M_i$

$$\tau_i(\tilde{x}) = \inf\left\{t \geq 0 : \Phi_i(t,\tilde{x}) \in \cup_{j=1}^{s_i} N_{ij}\right\}. \tag{9}$$

Furthermore define

$$\delta = \inf_{l\in L}\left(\inf\left\{\tau_l(\tilde{x}) : \tilde{x} \in \cup_{V(i,j)=l}\, g_{ij}(N_{ij})\right\}\right). \tag{10}$$

The number δ is positive in view of (3) and compactness. For $0 \leq t < \delta$ define a local semiflow by

$$\tilde{\Phi}(t,\tilde{x}) = \begin{cases} \Phi_i(t,\tilde{x}) & \text{if } \tilde{x} \in M_i \text{ and } 0 \leq t \leq \tau_i(\tilde{x}), \\ \Phi_l(t-\tau_i(\tilde{x}), g_{ij}(\Phi_i(\tau_i(\tilde{x}),\tilde{x}))) & \text{if } \tilde{x} \in M_i,\ \Phi_i(\tau_i(\tilde{x}),\tilde{x}) \in N_{ij}, \\ & V(i,j) = l, \text{ and } \tau_i(\tilde{x}) < t < \delta. \end{cases} \tag{11}$$

The function $\tilde{\Phi}$ is well-defined and satisfies

$$\tilde{\Phi}(t+s,\tilde{x}) = \tilde{\Phi}(s,\tilde{\Phi}(t,\tilde{x})), \quad \tilde{\Phi}(0,\tilde{x}) = \tilde{x} \tag{12}$$

for $t,s \geq 0$ with $t+s < \delta$. Since δ is uniform and $\tilde{\mathcal{M}}$ is compact, this local semiflow can be continued to a semiflow.

These properties carry over to Φ as defined by (8).

Corollary 4: *The trajectory* $t \mapsto \Phi(t,x)$ *is continuous (in the topology of* $\mathcal{M}$*). The trajectory* $t \mapsto \tilde{\Phi}(t,\tilde{x})$ *has at most one discontinuity (in the topology of* $\tilde{\mathcal{M}}$*) in any given time interval of length* δ.

The flow $\tilde{\Phi}$ can be recovered from Φ as

$$\tilde{\Phi}(t,\tilde{x}) = \lim_{\substack{s\to t\\ s<t}} \alpha(\Phi(s,x)), \quad \tilde{x} \in x, \quad \text{for} \quad t > 0,$$

$$\tilde{\Phi}(0,\tilde{x}) = \tilde{x}.$$

It is evident that both $\tilde{\Phi}(t,\tilde{x})$ and $\Phi(t,x)$ are not continuous in $\tilde{x}$ or x, respectively. There is no sensible way to make these functions continuous. In some sense this is the sacrifice one has to make in replacing smooth three-dimensional vector fields by vector fields on branched manifolds. Some continuity can be recovered under transversality assumptions.

The idea of the construction can be explained as follows. For a given nontrivial point $x \in \mathcal{M}$ there are several points $\tilde{x} \in M_i$ for appropriate $i \in L$. Thus there are several candidates $f_i(\tilde{x})$ for a tangent vector. By the principle of the actual point one of these tangent vectors is selected. Thus at every point of $\mathcal{M}$ there is a unique tangent vector. This vector field is piecewise smooth though it may not be smooth if $\mathcal{M}$ is embedded into some space of higher dimension.

The construction of Williams is somewhat different. Geometrically speaking, his construction assumes that the different manifolds glued together, embedded into some space of higher dimension, are tangent to each other.

Transversality

Suppose N is a closed C^1 curve in M_i. The vector field f_i is called transversal to N at $\tilde{x} \in N$ if $\tilde{x}$ is a simple point of N and $f_i(\tilde{x})$ is not tangent to N at $\tilde{x}$. The vector field f_i is called transversal to N if it is transversal to N at every point of N.
A point $\tilde{x} \in \tilde{\mathcal{M}}$ is called transversal if either $\tilde{x} \in M_i^!$ or $\tilde{x} \in N_{ij}$ and f_i is transversal to N_{ij} at $\tilde{x}$. The vector field $\tilde{f}$ is called transversal if all points of $\tilde{\mathcal{M}}$ are transversal.
A point $x \in \mathcal{M}$ is called transversal if all $\tilde{x} \in x$ are transversal.

Suppose $\Phi(t,x)$, $t \geq 0$, is a trajectory in $\mathcal{M}$. Suppose $\Phi(t_0,x)$ is transversal for some $t_0 > 0$. Then $\Phi(t,x)$ is transversal for $t_0 \leq t < t_0 + \delta$ where δ is defined by (9).

Theorem 5: *Suppose* $x \in \mathcal{M}$ *is trivial and* $\Phi(t,x)$ *is transversal for all* $t \geq 0$. *Then for every* $T > 0$ *there is a neighborhood* $U_T \subset \mathcal{M}$ *of* x *such that* Φ *is continuous in* $[0,T] \times U_T$.

Proof: The orbit $\Phi(t,x)$ through x has only countably many transitions $\tau_1 < \tau_2 < \cdots$. By assumption $\alpha(x) \in M_i^!$. Hence there is a neighborhood U_0 of x and a $\delta_0 > 0$, $\delta_0 < \delta$, such that $\Phi(t,y)$ contains only one point for $(t,y) \in [0,\delta_0) \times U_0$, and consequently Φ is continuous in this set.

Now suppose it has been shown that there is a δ_k, $0 < \delta_k < \delta$, and a neighborhood U_k of x such that $\Phi(t,y)$ is continuous in $[0,\tau_k+\delta_k) \times U_k$. Since $\Phi(\tau_k-\varepsilon,x)$ is transversal for small $\varepsilon > 0$ by assumption, by Ważewski's theorem [14] and assumption (3) there is a δ_{k+1}, $0 < \delta_{k+1} < \delta$, and a neighborhood $U_{k+1} \subset U_k$ such that $\Phi(t,y)$ is continuous in $[0,\tau_{k+1}+\delta_{k+1}) \times U_{k+1}$. This argument can be repeated.

We shall need a similar assertion for the flow $\tilde{\Phi}$.

Corollary 6: *Let* $\tilde{x} \in \tilde{\mathcal{M}}$, $\tilde{x} \in M_i^!$, *and suppose that* $\tilde{\Phi}(t,\tilde{x})$ *is transversal for all* $t \geq 0$. *Then for every* $T > 0$ *there is a neighborhood* $\tilde{U}_T \subset M_i$, $\tilde{U}_T \ni \tilde{x}$, *such that* $\tilde{\Phi}$ *is continuous in* $[0,t] \times \tilde{U}_T$.

The proof is essentially the same as that of Theorem 5.

The assumption of transversality everywhere is much too restrictive. In order to have something concrete at hand we define a class of vector fields which have some generic properties.

Suppose $x \in \mathcal{M}$ is a branch point which is not transversal. Then there is $\tilde{x} \in x$ such that $\tilde{x} \in N_{ij} \subset M_i$, and f_i is tangent to N_{ij}. We call a branch point x a generic contact point if for all $\tilde{x} \in x$ the following is true: If $\tilde{x} \in N_{ij}$ then the contact of $\Phi_i(t,x)$ and N_{ij} is only of first order. Then locally $\Phi_i(t,\tilde{x})$ stays on one side of N_{ij} (see Fig.4).

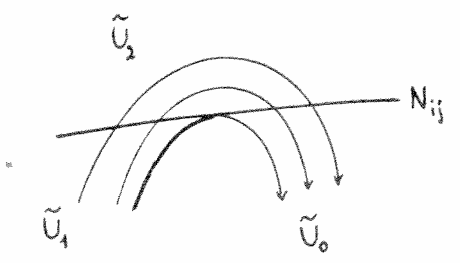


Fig.4: Generic contact point

If x is a generic contact point then at each $\tilde{x} \in x$, $\tilde{x} \in N_{ij}$, there is a neighborhood $\tilde{U} \subset M_i$ of $\tilde{x}$ such that the trajectory $\Phi_i(t,\tilde{x})$ and N_{ij} define three domains $\tilde{U}_0$, $\tilde{U}_1$, $\tilde{U}_2$ (see Fig.4). Trajectories of $\tilde{\Phi}$ starting in $\tilde{U}_0$ stay in M_i as long as they stay in $\tilde{U}$, trajectories in $\tilde{U}_1$ and $\tilde{U}_2$ leave M_i. That is why $\tilde{\Phi}$ is not continuous at these points.

We call a vector field f on $\tilde{\mathcal{M}}$ a field of generic contact if there are only finitely many points which are not transversal and if all these points are generic contact points. The vector fields in [6] have this property.

At least in topologically simple cases strong transversality properties provide equivalences between branched manifolds and unbranched manifolds of known topological structure. In the following examples 4a,b,c we assume that the vector field f_i is transversal to all curves N_{ij} and also to all curves $g_{jl}(N_{jl})$ with $V(j,l) = i$. We say that the Poincaré-Bendixson property holds for a trajectory when the trajectory eventually remains in some S^2 (or disc).

Example 4a: M is a 2-sphere, N and $g(N)$ are disjoint circles. These curves define an annulus A and two discs D_0 (bounded by N) and D_1 (bounded by $g(N)$). Since the vector field is transversal on N and $g(N)$ there are just four qualitatively different situations which can be presented as follows.
a) D_0 and D_1 are positively invariant. Then a trajectory either stays in D_0 or D_1, or it stays in A, or it leaves A to stay in D_1. Hence the Poincaré-Bendixson property holds.

b) Trajectories from the annulus never arrive at N. There is no reset. The Poincaré-Bendixson property holds.
c) D_0 is positively invariant, but D_1 is negatively invariant. Then the two discs can be discarded. The two curves can be identified, a 2-torus remains.

Example 4b: M_1, M_2 are 2-spheres, N_{11}, N_{21} and their images are disjoint circles. On M_i these curves define an annulus and two discs. There are 16 qualitatively different cases. If we exchange M_1 and M_2 then ten cases remain. Among these all cases are trivial where either N_{11} or N_{21} cannot be reached from the interior of the annulus. In these cases every trajectory has at most one transition from one sphere to the other. Then four cases remain. In three of these cases one sees easily, as in Example 1, that the Poincaré-Bendixson property holds. In the remaining case one can disregard the four discs, and connect the two annuli to a 2-torus. Thus we conclude that the dynamics of a transversal vector field on this branched manifold can be essentially represented on a 2-sphere or on a 2-torus.

Example 4c: M_1, M_2 are 2-spheres with $s_1 = 2$ and $s_2 = 1$. The six resulting curves and the direction of the transversal vector field are shown in Fig.5. This structure cannot be reduced to a smooth (classical) manifold. A similar observation holds for the Lorenz manifold of Williams [16], [17].

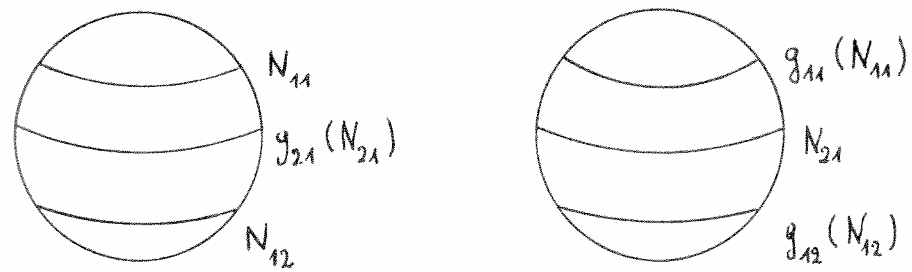


Fig.5: Illustration of Example 4c

Limit sets

As usual the limit set (ω limit set) of a trajectory $\Phi(t,x)$ is defined as

$$\omega(x) = \{z : \exists t_1 < t_2 < \cdots,\ t_k \to \infty,\ \Phi(t_k,x) \to z\}.$$

As in the classical case one proves the following proposition.

Proposition 7: *The set $\omega(x)$ is*
i) nonempty,
ii) compact in the topology of $\mathcal{M}$,
iii) connected in the topology of $\mathcal{M}$.

Proof:
i) Choose any sequence t_k. $\mathcal{M}$ is compact. Hence there is an accumulation point.

ii) Let $y \in \mathcal{M}$, $y \notin \omega(x)$. Then there is a neighborhood U of y and a $T > 0$ such that $U \cap \{\Phi(t,x) : t \geq T\} = \emptyset$. Hence $\omega(x)$ is closed and thus compact.
iii) Suppose $\omega(x)$ is not connected in the topology of $\mathcal{M}$. Then there are open sets U and V such that $U \cap V = \emptyset$, $\omega(x) \subset U \cup V$, and $\omega(x) \cap U \neq \emptyset$, $\omega(x) \cap V \neq \emptyset$. Define $K = \mathcal{M} \setminus (U \cup V)$. K is compact. For every $T > 0$ there are $t_1 > T$, $t_2 > T$ such that $\Phi(t_1, x) \in U$, $\Phi(t_2, x) \in V$, hence also $t > T$ such that $\Phi(t,x) \in K$. Hence there is a sequence $t_k \to \infty$ such that $\Phi(t_k, x)$ converges to some point $y \in K$. Hence $y \in \omega(x)$. This gives a contradiction.

Limit sets on branched manifolds need not be positively invariant. A simple counterexample: In Example 1 above assume that a trajectory in $M \setminus N$ approaches a stationary point on N. Then the stationary point is the limit set, but if the stationary point is chosen as an initial condition then the trajectory continues on $g(N)$.

Even if the vector fields f_i are structurally stable (see, e.g., [10]) and all critical elements are transversal to the N_{ij} there may be limit sets which are not positively invariant. This fact is shown by Example 5b. However, one can prove the following result.

Theorem 8: *Assume that all points of $\omega(x)$ are transversal. Then $\omega(x)$ is positively invariant.*

Proof: Let $y \in \omega(x)$. By assumption there is a sequence $t_k \to \infty$ such that $\Phi(t_k, x) \to y$. We have to show $\Phi(s,y) \in \omega(x)$ for $s \geq 0$. We can use the flow property

$$\Phi(t_k + s, x) = \Phi(s, \Phi(t_k, x)).$$

If Φ were continuous in x then $\Phi(t_k + s, x) \to \Phi(s,y)$ would follow immediately. Since Φ is not continuous in general, we have to consider the problem in more detail.

Case 1: y is trivial. In view of Theorem 5, for every $T > 0$, there is a neighborhood $U_T \subset \mathcal{M}$, $U_T \ni y$, such that Φ is continuous in $[0,T] \times U_T$. Choose $T > s$. For large k we have $\Phi(t_k, x) \in U_T$. We use

$$\Phi(s + t_k, x) = \Phi(s, \Phi(t_k, x)).$$

For $k \to \infty$ the right hand side has the limit $\Phi(s,y)$ whereas the left hand side, by definition of the limit set, coverges to some point in $\omega(x)$.

Case 2: y is not trivial. Let $\alpha(y) = \tilde{y} \in M_l$ be the actual point. The sequence $\Phi(t_k, x)$ may contain nontrivial points.

Case 2a: There is an infinite subsequence such that the actual points are in M_l. We can assume that the given sequence has already this property, i.e., that $\alpha(\Phi(t_k, x)) = \tilde{y}_k \in M_l$. By Corollary 6, for every $T > 0$ there is a neighborhood $\tilde{U}_T \subset M_l$, $\tilde{U}_T \ni \tilde{y}$ such that $\tilde{\Phi}$ is continuous in $[0,T] \times \tilde{U}_T$. Choose $T > s$. For large k we have $\tilde{\Phi}(t_k, \alpha(x)) \in \tilde{U}_T$. Then

$$\tilde{\Phi}(s + t_k, \alpha(x)) = \tilde{\Phi}(s, \tilde{\Phi}(t_k, \alpha(x)),$$

and thus

$$\Phi(s + t_k, x) = \Phi(s, \Phi(t_k, x)).$$

For $k \to \infty$ we have $\tilde{\Phi}(t_k, \alpha(x)) \to \tilde{y}$, thus $\Phi(t_k, x) \to y$. Now we can continue as in Case 1.
Case 2b: There is no such sequence as in Case 2a. We choose, if necessary, a subsequence, and have the following situation. There is $\hat{y} \in y$, $\hat{y} \neq \alpha(y)$, $\hat{y} \in N_{ij} \subset M_i$. There is an open neighborhood $U \subset M_i$ of $\hat{y}$ such that $N_{ij} \cap U$ is homeomorphic to an interval, and $N_{ij} \cap U$ separates U into two open sets U_- (where trajectories of $\tilde{\Phi}$ leave M_i) and U_+ (where trajectories of $\tilde{\Phi}$ stay in M_i) and $U = U_+ \cup U_- \cup (N_{ij} \cap U)$. We can assume that Φ_i is transversal to N_{ij} along $N_{ij} \cap U$.
Case 2bα: There is an infinite subsequence such that the actual points are in U_+. We can assume that the original sequence has this property, i.e., $\alpha(\Phi(t_k, x)) \in U_+$. For large k the trajectory of Φ_i (or $\tilde{\Phi}$ through $\alpha(\Phi(t_k, x))$ crosses N_{ij} close to $\hat{y}$. In U_+ this trajectory is a trajectory of Φ. Hence for each k there is an s_k such that $\alpha(\Phi(t_k - t, x)) \in U_+$ for $0 \leq t < s_k$, and $\Phi_i(-s_k, \alpha(\Phi(t_k, x)) \in N_{ij}$. This is a contradiction to the fact that there is a transition to M_l at N_{ij}. Hence Case 2bα is impossible.
Case 2bβ: There is an infinite subsequence such that $\alpha(\Phi(t_k, x)) \in U_-$. We can assume that the given sequence has this property. Choose $\varepsilon > 0$ such that $\Phi_i(t, \hat{y}) \in U_-$ for $-\varepsilon \leq t < 0$. Then for k sufficiently large there is s_k such that $\Phi(t_k + s_k + t, x) \in U_-$ for $-\varepsilon \leq t < 0$. Then $\Phi(t_k + s_k + t, x) \to \Phi_i(t, \tilde{y})$ uniformly in $-\varepsilon \leq t < 0$ for $k \to \infty$. Hence $\Phi_i(t, \tilde{y}) \in \omega(x)$ for $-\varepsilon \leq t < 0$. Now choose any of these points and proceed as in Case 1.

We add some comments on attractors and basins. Consider a vector field on S^2 with finitely many critical elements. Then the basins of the attractors form finitely many open domains and their boundaries are formed by trajectories, hence are piecewise differentiable curves. These boundaries may contain repellers and saddle points. The basin boundaries are themselves invariant sets. On branched manifolds we have a different situation. In Example 5b there are finitely many stationary points and periodic orbits, every trajectory converges to one of these. The basin of each attractor contains an open set such that the closures of these sets cover the whole manifold. But the basins need not be open.

Return mappings and global behavior

The global behavior of the dynamical system defined by $(\mathcal{M}, f)$ can be studied by return mappings (Poincaré mappings). The idea is that trajectories which have only finitely many transitions stay eventually in one of the M_i and hence are "trivial". Thus one chooses a curve L (preferably one of the curves N_{ij} or $g_{ij}(N_{ij})$) and one follows all trajectories starting from L. For $x \in L$ (in the following we shall omit all tildes) define

$$\tau(x) = \inf\{t > 0 : \Phi(t,x) \in L\}.$$

Then define

$$L_\infty = \{x \in L : \tau(x) = \infty\}.$$

Introduce the symbol $\emptyset$ for "empty". Define a mapping φ on $L \cup \{\emptyset\}$ by

$$\varphi(x) = \begin{cases} \Phi(\tau(x), x) & \text{if } x \in L \setminus L_\infty, \\ \emptyset & \text{if } x \in L_\infty, \\ \emptyset & \text{if } x = \emptyset. \end{cases}$$

The mapping φ is called the return mapping. At least in cases where the number of the N_{ij} is small one can get a global view of the asymptotic behavior by studying φ and its iterates.

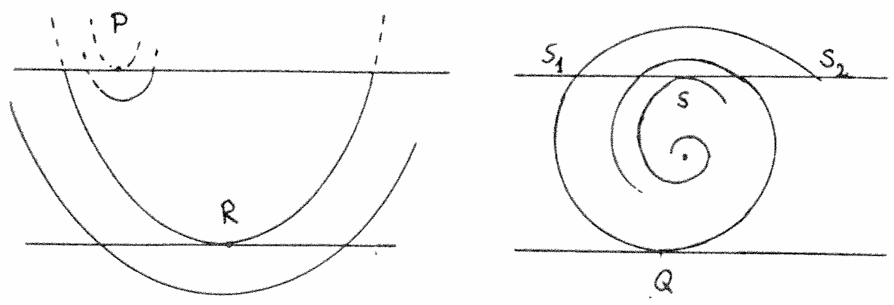
If x is such that $\varphi^k(x) = \emptyset$ for some k then the trajectory $\Phi(t,x)$ meets L only finitely often. Fixed points of φ other than $\emptyset$ correspond to periodic orbits.

Here we study a type of branched manifold closely related to Example 3. Let $M_1 = \mathbb{R}^2$, $M_2 = \mathbb{R}^2$, each endowed with the same cartesian coordinate system (x,y). Let $N_{11} = \{x = b\}$ and $N_{21} = \{x = a\}$, and let $g_{11}(x) = x$, $g_{21}(x) = x$ (with respect to the identical coordinate system). For convenience we assume $a < b$. If we discard the sets $\{x \in M_1 : x > b\}$ and $\{x \in M_2 : x < a\}$ then we arrive at the manifold studied in [6].

As in [6] we introduce the sets $\mathcal{A}_1 = \{(x,y) \in M_1 : a < y < b\}$, $\mathcal{A}_2 = \{(x,y) \in M_2 : a < y < b\}$, $\mathcal{B} = \{(x,y) \in M_2 : y > b\}$, $\mathcal{C} = \{(x,y) \in M_1 : y < a\}$. Furthermore we introduce the lines $L_a = \{y = a\}$ and $L_b = \{y = b\}$ as subsets of M_1 and M_2. It is sufficient to consider trajectories which meet N_{11} and N_{21} infinitely often. Choose $L = L_a$. Define a mapping φ_1 as follows. For any point $x \in L_a \subset M_1$ define $\tau_1(x) = \inf\{t > 0 : \Phi(t,x) \in L_b \subset M_1\}$. Define $\varphi_1(x) = \Phi(\tau_1(x), x)$ if $\tau_1(x)$ is finite and $\varphi_1(x) = \emptyset$ otherwise. Extend the definition of φ_1 by putting $\varphi_1(\emptyset) = \emptyset$. Define τ_2 and φ_2 using M_2 instead of M_1. Then $\varphi = \varphi_2 \circ \varphi_1$ is the return mapping.

To have something concrete at hand we assume that the trajectories in $\mathcal{A}_1 \cup \mathcal{C}$ look as in Fig.6a, whereas the trajectories in $\mathcal{A}_2 \cup \mathcal{B}$ look as in Fig 6b.

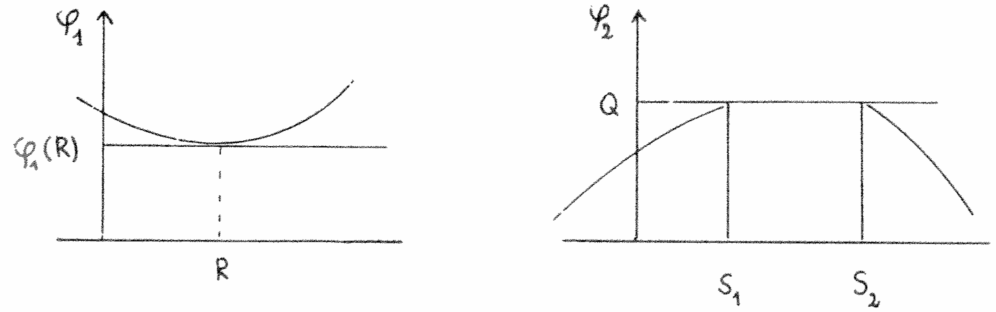
Apparently there are four interesting points called P, Q, R, S as indicated. We assume that P and S do not coincide on $\mathcal{M}$, neither do Q and R. The points P and Q are generic contact points. The points R and S are just points where f_l is not transversal to $g_{ij}(N_{ij})$.



a) b)

Fig.6: The flows on M_1 and M_2

Then the function φ_1 is continuous everywhere, it is decreasing for $x < R$ and increasing for $x > R$. Hence it attains its minimum at R (see Fig 7a). Let S_1, S_2 be the two preimages of Q with respect to the flow in $\mathcal{A}_2$. Then φ_2 increases for $x < S_1$, decreases for $x > S_2$, and φ_2 takes the interval (S_1, S_2) to $\emptyset$ (see Fig.7b).



a) b)

Fig.7: The maps φ_1 and φ_2

We have to study the composition $\varphi_2 \circ \varphi_1$. The behavior depends on the relative position of some interesting points. For $x \ll 0$ the function φ_1 is large and decreasing, thus φ is small and increasing. For $x \gg 0$ the function φ_1 is large and increasing, thus φ is large and decreasing. We assume that φ is dissipative in the sense that $\varphi(x) < x$ for $x \gg 0$ and $\varphi(x) > x$ for $x \ll 0$.

The minimum $\varphi_1(R)$ of φ may be located below S_1, between S_1 and S_2, or above S_2 (we do not discuss limit cases).

Case 1: $\varphi_1(R) > S_2$. Then the function φ does not assume the value $\emptyset$, it is first increasing, then decreasing. The maximum is assumed at R. If $\varphi(R) < R$ then there is an odd number of fixed points, all located below R, these are alternatingly stable and unstable, every trajectory (of φ) converges to one of these fixed points. If $\varphi(R) > R$ then there is an even number of fixed points in $(-\infty, R]$, and a

single fixed point in $[R, +\infty)$. The latter may be unstable and give rise to period doubling or other complex behavior.
Case 2: $\varphi_1(R) \in (S_1, S_2)$. There are two values R_2 and R_2' with $R_2 < R < R_2'$ and $\varphi_1(R_2) = \varphi_1(R_2') = S_2$. The function φ is increasing in $(-\infty, R_2]$, decreasing in $[R_2', +\infty)$, and it carries (R_2, R_2') into $\emptyset$. Furthermore, $\varphi(R_2) = \varphi(R_2') = Q$. Now there are again three cases.
If $Q < R_2$ then there is an odd number of fixed points in $(-\infty, R_2]$, and no fixed point in $[R_2', +\infty)$. Trajectories cannot enter the interval (R_2, R_2'), all trajectories end up in $(-\infty, R_2]$, and approach one of the fixed points.
There is a largest fixed point in $(-\infty, R_2]$, and this fixed point is stable (in a generic situation, otherwise it is stable from above). This fixed point corresponds to a stable periodic orbit of Φ. The interval (R_2, R_2') corresponds to trajectories of Φ which approach the attractor in $\mathcal{A}_2 \cup \mathcal{B}$. Trajectories of φ cannot enter (R_2, R_2'). The trajectory of Φ starting from R_2 is the boundary between the basins of the stable periodic orbit and the attractor in $\mathcal{A}_2 \cup \mathcal{B}$. The boundary itself approaches the attractor.
If $Q \in (R_2, R_2')$ then there is an even number of fixed points in $(-\infty, R_2]$, and no fixed point in $[R_2', +\infty)$. Some trajectories (of φ) can end up in (R_2, R_2') (all trajectories will end up in this interval, if φ has no fixed points.
If $Q > R_2'$ then there is an even number of fixed points in $(-\infty, R_2]$, and exactly one fixed point in $[R_2', +\infty)$. The interval (R_2, R_2') will attract some trajectories.

Case 3: $\varphi_1(R) < S_1$. There are four values $R_2 < R_1 < R < R_1' < R_2'$ such that $\varphi_1(R_2) = \varphi_1(R_2') = S_2$, $\varphi(R_1) = \varphi(R_1') = S_1$. There are five cases depending on where the point Q is located in relation to these numbers. We shall list some essential features.
$Q < R_2$. In $(-\infty, R_2]$ an odd number of fixed points, no fixed points otherwise.
$R_2 < Q < R_1$. In $(-\infty, R_2]$ there is an even number of fixed points, no fixed points otherwise.
$R_1 < Q < R_1'$. An even number of fixed points in $(-\infty, R_2]$, an odd number of fixed points in (R_1, R_1'), no fixed point in $[R_2', +\infty)$.
$R_1' < Q < R_2'$. An even number of fixed points in $(-\infty, R_2]$ and in (R_1, R_1'), no other fixed points.
$R_2' < Q$. An even numer of fixed points in $(-\infty, R_2]$ and in (R_1, R_1'), exactly one fixed point in $[R_2', +\infty)$.

It should be underlined that the return map φ descr ibes all trajectories $\Phi(t, x)$ of the original system which meet the curve L_a. All other trajectories have trivial behavior insofar as they stay eventually either in M_1 or in M_2.

Theorem 9: *Under the assumptions stated above there are two types of limit sets: limit sets of the Poincaré-Bendixson type in $\mathcal{A}_0 \cup \mathcal{C}$ and $\mathcal{A}_1 \cup \mathcal{B}$, and periodic orbits which meet L_a and L_a.*

Numerical examples

On the manifolds M_i of the preceding section consider two vector fields f_i, in polar coordinates,

$$f_i : \quad \begin{aligned} \dot{r} &= r(1 - r/R_i), \\ \dot{\varphi} &= 1. \end{aligned}$$

where $i = 1, 2$. Hence the problem depends on four parameters a, b, R_0, R_1. In the numerical study we keep R_1, R_2 fixed and vary a, b. The numerical calculations indicate that all non-constant solutions converge to periodic solutions. The number of periodic solutions varies between one and two (and not between one and three, as one might think). The transition between the different cases shows the saddle-node bifurcation of periodic orbits which has been found in [6].

Example 5: Let $r = 2$, $s_1 = 1$, $s_2 = 1$, $V(1,1) = 2$, $V(2,1) = 1$, $M_1 = S^2 = \mathbb{R}^2 \cup \{\infty\}$, $M_2 = S^2 = \mathbb{R}^2 \cup \{\infty\}$. Define the g_{ij} with the obvious identifications in cartesian coordinates
$N_{11} = \{(x, y) : y = b\}$, $g_{11} : N_{11} \to M_2$, $x \mapsto x$,
$N_{21} = \{(x, y) : y = a\}$, $g_{21} : N_{21} \to M_1$, $x \mapsto x$.
a) $R_1 = 1$, $R_2 = 0.5$. $a = -0.4$, and b is ranging from 0.1 to 0.9. Then there are two unstable stationary points on $\mathcal{M}$, and the two circles are not periodic orbits on $\mathcal{M}$. Numerical evidence shows that there is a unique "large" periodic orbit, which is globally stable (with the exception of the stationary points). Fig.8 shows the case $b = 0.4$.

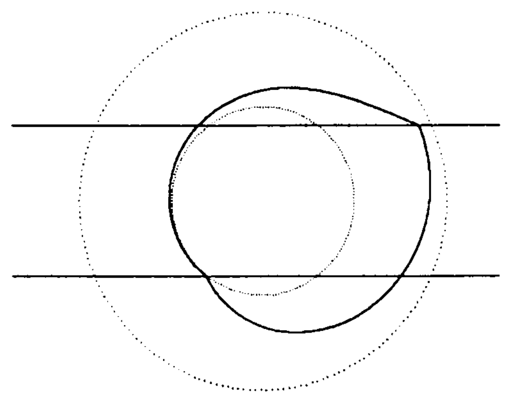


Fig.8: One attracting periodic orbit

b) $R_1 = 1$, $R_2 = 0.3$, $b = 0.2$, and a ranges from -0.4 to -0.1. There are two unstable stationary points on $\mathcal{M}$, and the smaller circle is a locally stable periodic orbit on $\mathcal{M}$ for $-0.4 \le a < 0.3$.

For $-0.4 \le a \le -0.3$ the smaller circle is a limit set. But for $a = -0.3$ the smaller circle is not a periodic orbit. The trajectory starting at $x = 0$, $y = -0.3$ leaves the limit set. Hence the limit set is not positively invariant.

Numerical evidence shows that the smaller circle is globally stable (with the exception of the stationary points) for a close to -0.4. On the other hand, for $a \in [-0.3, -0.1)$ we are in the situation of Example 5a, there is a single "large" periodic orbit which changes between M_1 and M_2.

However, for a in between something interesting happens. At $a = \hat{a} \approx -0.302$ there is a saddle-node bifurcation of periodic orbits which results in a "large" stable periodic orbit and a basin boundary. For $a \in (\hat{a}, -0.3)$ the two stable orbits coexist. The boundary of the basins is defined by the trajectory of f_2 which is tangent to the line L_a. The boundary trajectory itself approaches the large periodic orbit. Hence the basin is not open. Fig.9 shows the situation for $a = -0.301$.

c) $R_1 = 1$, $R_2 = 0.2$, $a = 0.5$, $b = 0.7$. There are two unstable stationary points and one stable periodic orbit which shows a complicated behavior (Fig.10).

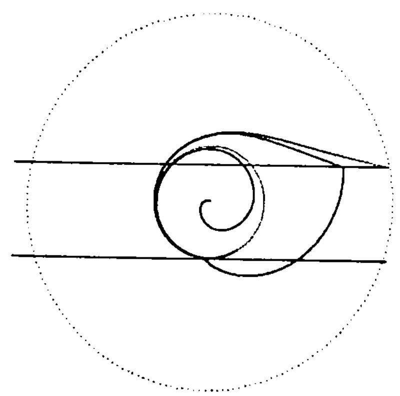


Fig.9: Coexistence of two stable periodic orbits

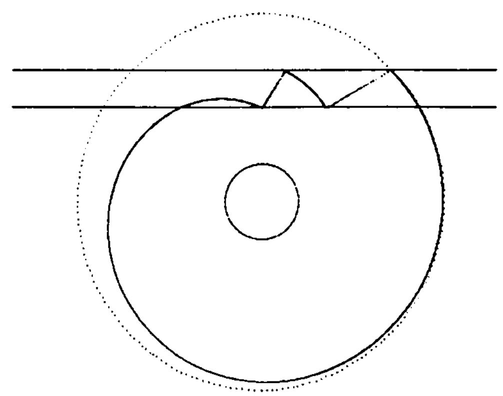


Fig.10: Complicated orbit

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