

Math 372, Mathematical Modelling
Fall 2010

Assignment 2, due Oct 08, 12:00 in class.

Exercise 6:

[3]

Use a computer software package, such as Maple or Matlab and plot the following two functions and their difference

$$f(x) = \frac{1}{1 + e^{-2x}}, \quad g(x) = \frac{1}{2}(1 + \tanh(x)).$$

The result should make you suspicious. Can you explain what's happened?

Exercise 7:

[3]

A liquor transfer company needs to store wine and beer in a cooled storage house. This cooling house has a limited capacity of $T = 24ft^3$. The costs for storage of the alcoholic products is given

$$C(x_1, x_2) = \left(\frac{a_1 b_1}{x_1} + \frac{h_1 x_1}{2} \right) + \left(\frac{a_2 b_2}{x_2} + \frac{h_2 x_2}{2} \right),$$

where x_i denotes the amount of beverage of type 1 (beer) and type 2 (wine), a_i denotes the cost of the beverage, b_i is the withdrawal rate per unit of time, and h_i denotes the storage cost per unit of beverage of type i . Let t_i denote the used space in cubic feet to store a unit of the corresponding beverage. Then the total space is filled if

$$g(x_1, x_2) = t_1 x_1 + t_2 x_2 = T.$$

1. Minimize the costs for the following parameter values:

$$a_1 = 9, a_2 = 4, b_1 = 3, b_2 = 5, h_1 = 0.5, h_2 = 0.2, t_1 = 2, t_2 = 4.$$

2. Find the shadow prize and explain its meaning for the liquor transfer company.

Exercise 8:

[5]

Consider the model

$$a_{n+1} = a_n + k(M - a_n)(a_n - m),$$

with $M, m, k > 0$, where a_n represents the number of individuals in a whale population after n years.

- A- Discuss the model. What is the meaning of each of the model parameters?
- B- Let $M = 5000$, $m = 100$, and $k = 0.0001$. Find the fixed points of the model. Determine the stability of the fixed points using linear stability analysis, and check the results using graphical (cobwebbing) analysis.
- C- Try several different values for the initial condition a_0 . Are your simulations consistent with your stability analysis from part (b)? As part of your answer, submit a graph of a_n versus n for some well-chosen initial conditions, and briefly describe your results in terms of the whale population.

- D- The model has two serious shortcomings. What are they (hint: what happens when $a_0 < m$, and when $a_0 \gg M$)? Think of a possible way to fix the model so as to overcome these shortcomings. You do *not* need to write down the equation of an improved model; it is sufficient to describe your ideas with words and/or sketches of graphs (hint: it helps to study the behaviour of the given model with a sketch of the graph of a_{n+1} versus a_n).

Exercise 9:

[4]

You might know that the entropy is used to measure the degree of order or disorder in a physical system. In this exercise, we wish to find a discrete probability distribution $(p_1, \dots, p_n)$, which maximizes the entropy, i.e. which has the least information (most disorder). The entropy is given by

$$E(p_1, \dots, p_n) = - \sum_{k=1}^n p_k \ln p_k,$$

and the constraint is the fact that the p_i form a probability distribution, i.e.

$$\sum_{k=1}^n p_k = 1.$$

Find the distribution with the greatest entropy (i.e. find a formula for the values of $p_i, i = 1, \dots, n$).

Exercise 10:

[5]

We consider selection in a diploid population, where the trait of interest resides on a single locus with the alleles W and w. This means a given individual can have one of three genotypes: WW, Ww, or ww. These genotypes might refer to different phenotypes, for example blue eyes, brown eyes and grey eyes, respectively. Here we study two phenotypes for moth's wing color: WW: black wings and ww: white wings.

We can derive a recursive relation between the frequency of the W allele, p_n , in a population from one generation to the next:

$$p_{n+1} = \frac{\alpha p_n}{(\gamma - \alpha)p_n^2 - 2(\gamma - \alpha)p_n + \gamma},$$

where $\alpha > 0$ and $\gamma > 0$ denote the fitness of black and white wings, respectively. For example $\alpha > \gamma$ indicates a selective advantage of the black winged moths over the white winged moth.

1. Find the fixedpoints of the above discrete dynamical system and discuss their stability via linearization.
2. Use cobwebbing to decide in which case do the black moths dominate and in which case do the white moths dominate.