

Math 372, Mathematical Modelling
Fall 2010

Assignment 1, due Sep 24, 12:00 in class.

Exercise 1:

[3]

You are arriving late from an international flight at Toronto Airport and you need to hurry to get to the gate of the connecting flight. We assume that the distance between your arrival gate and the departure gate is 900 m of which 300 m are covered by human conveyor bands (flat escalators). You try to get to the connecting gate as soon as possible, but you have two restrictions:

- You will have to stop once to tie your shoe laces.
 - You have a quite heavy duffle bag and you can only run for 200 m in total. The rest you walk.
1. Find realistic speed for walking, running with luggage, and for the human conveyor bands.
 2. Under the above conditions, is it faster to tie your laces while on the conveyor belt or off the belt? Is it more efficient to run on the conveyor belt or off the belt?
 3. Try to formulate a general principle based on your results.

Exercise 2:

[3]

Assume you buy a house and you accept a loan of \$250,000 with an amortization period of 25 years and an average annual interest rate of 4.5 %.

1. What is your monthly payment?
2. Plot and compute the sensitivity of the monthly payment on the interest rate.

Exercise 3:

[4]

Water enters a cylindrical tank of cross-sectional area A at a constant rate (volume per unit time). The tank has a small hole in the bottom, and so water flows out through the hole. Let $h(t)$ be the depth of the water in the tank at time t . A version of Bernoulli's Principle in hydrology says that the square of the rate of flow out of the tank is proportional to h .

1. Develop a differential equation for h . Give a physical interpretation of all model parameters you introduce. You can assume that the tank is standing upright, that is, its footprint is circular.
2. Find the steady states, h^* . Give a physical interpretation of the steady states, and discuss the dependence of h^* on the model parameters (do the relationships make sense?).
3. Determine the stability of the steady states two different ways (graphically as well as with a linear stability analysis). What is the problem at $h^* = 0$?

Exercise 7:

[5]

Consider a sky diver in free fall. During free fall, the forces acting on a sky diver are gravity and air resistance. Gravity causes acceleration, and air resistance causes deceleration.

1. Apply Newton's second law of motion to the sky diver (the mass m of the sky diver times her acceleration is equal to the sum of the forces acting on her), and derive a differential equation for her velocity at time t , $v(t)$. Recall that the force due to gravity is mg , where $g = 32 \text{ ft/sec}^2$. Experiments have shown that the force of air resistance can be modelled fairly well by a term that is proportional to velocity.
2. Obtain the analytical solution for $v(t)$, assuming $v(0) = 0$.
3. Find the limit of $v(t)$ as $t \rightarrow \infty$. This limiting velocity is known as the *terminal velocity*.
4. Give a graphical stability analysis of the differential equation, and thereby re-derive a formula for the terminal velocity. Give a physical interpretation of the terminal velocity in terms of the forces acting on the sky diver.

Exercise 5:

[5]

The *Gompertz* model is used to describe a growing tumor. Let $u(t)$ denote the tumor size at time t , then

$$\dot{u} = -ru \ln\left(\frac{u}{\theta}\right).$$

1. Sketch the graph of the right hand side. First try with pen and paper. If this doesn't work then plot it with some software.
2. Give an interpretation of the model and of the parameters r and θ .
3. Find the steady states and determine their stability. For that we might introduce the notion of "super" unstable, if $f'(u^*) = \infty$.
4. Sketch typical solutions in the phase plane $(u, f(u))$ and as time dependent solution $(t, u(t))$. Give an interpretation of these results.
5. If you want to earn 2 extra points then you can use separation to find an explicit solution of the Gompertz model.