

THE SLEEP PROBLEM

Suppose that after many years of observation, you notice that your sleep is self-correcting: The amount of sleep you get one night is the average of the amount of sleep you got on the previous two nights. On two particular nights (initial conditions), you got 7 hours of sleep, followed by 8 hours of sleep. How much sleep will you get n nights later? How much sleep will you get each night in the long run if this pattern continues? Are the results the same if the initial conditions are reversed (on two particular nights, you got 8 hours of sleep, followed by 7 hours of sleep)?

Solution:

Let H_m be the amount of sleep (measured in hours) you get on the m -th night (we will worry about any offset induced by the phrase " n nights *later*" at the end of the problem).

Then our model is

$$\begin{aligned}H_1 &= 7, \\H_2 &= 8, \\H_m &= \frac{H_{m-1} + H_{m-2}}{2}, \text{ for } m = 3, 4, \dots\end{aligned}$$

Note: You may have labelled the initial conditions H_0 and H_1 , and/or written the general model as $H_{m+1} = \frac{H_m + H_{m-1}}{2}$. Keep this in mind when comparing your solution to the one provided here!

Our model is a second-order linear equation. To solve (find an explicit expression for H_m in terms of m), we try $H_m = p^m$, where p is to be determined.

Substituting $H_m = p^m$ into the model gives

$$p^m = \frac{1}{2}(p^{m-1} + p^{m-2}).$$

Divide by the common factor p^{m-2} to get

$$p^2 = \frac{1}{2}(p + 1),$$

or

$$p^2 - \frac{1}{2}p - \frac{1}{2} = 0,$$

which has roots $p = 1$ and $p = -1/2$.

We have found two linearly independent solutions, namely

$$\begin{aligned}H_m &= 1^m = 1, \\H_m &= \left(-\frac{1}{2}\right)^m.\end{aligned}$$

Any linear combination of these two solutions also is a solution. Thus, the general solution for our model is

$$H_m = A(1)^m + B\left(-\frac{1}{2}\right)^m = A + B\left(-\frac{1}{2}\right)^m,$$

where A and $B \in \mathcal{R}$.

We now use the initial conditions $H_1 = 7$ and $H_2 = 8$ to determine A and B :

$$\begin{aligned} H_1 &= A - \frac{1}{2}B = 7, \\ H_2 &= A + \frac{1}{4}B = 8, \end{aligned}$$

giving $A = 23/3$ and $B = 4/3$, so that the solution is

$$H_m = \frac{23}{3} + \frac{4}{3}\left(-\frac{1}{2}\right)^m, \text{ for } m = 3, 4, \dots$$

Note: Your solution may look different on first glance, but if your H_3 works out to 7.5, it's probably correct.

Now, how much sleep would you get n nights *later* (following the first night of 7 hours, and the second night of 8 hours)? We need an offset (you will not need an offset if you set up your model with initial conditions H_{-1} and H_0 , and have a general formula for H_n). In particular, the amount of sleep obtained n nights *later* is

$$H_{n+2} = \frac{23}{3} + \frac{4}{3}\left(-\frac{1}{2}\right)^{n+2} = \frac{23}{3} + \frac{1}{3}\left(-\frac{1}{2}\right)^n, \text{ for } n = 1, 2, \dots$$

Note: The right-hand side of this formula should be the same for everyone, no matter how you set up the model and what you label the left-hand side.

How much sleep would you get each night in the long run if this pattern continues? We are interested in the limit of H_{n+2} as $n \rightarrow \infty$, which is $23/3$ hours of sleep.

Last but not least, if the initial conditions are reversed, the answer is different! In particular, with $H_1 = 8$ and $H_2 = 7$, we get

$$\begin{aligned} H_1 &= A - \frac{1}{2}B = 8, \\ H_2 &= A + \frac{1}{4}B = 7, \end{aligned}$$

giving $A = 22/3$ and $B = -4/3$, so that the solution is

$$H_m = \frac{22}{3} - \frac{4}{3}\left(-\frac{1}{2}\right)^m, \text{ for } m = 3, 4, \dots,$$

and

$$H_{n+2} = \frac{22}{3} - \frac{1}{3}\left(-\frac{1}{2}\right)^n, \text{ for } n = 1, 2, \dots$$

Although the third night of sleep still works out to 7.5 hours, in the long run, you now get $1/3$ hour less sleep, namely $22/3$ hours.