

Assignment 2 Solutions

Exercise 6: $f(x)$ and $g(x)$ are indeed the same functions.

If you plot their difference you might see a chaotic pattern which comes from numerical rounding errors.

To see that they are identical, we use one of

Euler's formulas: $\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$$\begin{aligned} g(x) &= \frac{1}{2} (1 + \tanh(x)) = \frac{1}{2} \left(1 + \frac{e^x - e^{-x}}{e^x + e^{-x}} \right) = \frac{1}{2} \left(\frac{e^x + e^{-x} + e^x - e^{-x}}{e^x + e^{-x}} \right) \\ &= \frac{1}{2} \cdot \frac{e^x}{e^x + e^{-x}} = \frac{1}{1 + e^{-2x}} = f(x) \end{aligned}$$

Exercise 7: see exercise 7.pdf

Exercise 8: see exercise 8.pdf

Exercise 9: $E(p_1, \dots, p_n) = - \sum_{k=1}^n p_k \ln p_k$

$$g(p_1, \dots, p_n) = \sum_{k=1}^n p_k = 1$$

$$\nabla E = \left(\frac{\partial E}{\partial p_1}, \frac{\partial E}{\partial p_2}, \dots, \frac{\partial E}{\partial p_n} \right) = (-\ln p_1 - 1, -\ln p_2 - 1, \dots, -\ln p_n - 1)$$

$$\nabla g = \left(\frac{\partial g}{\partial p_1}, \dots, \frac{\partial g}{\partial p_n} \right) = (1, \dots, 1)$$

$$\text{Lagrange multiplier: } \left. \begin{array}{l} \nabla E = \lambda \nabla g \\ g = 1 \end{array} \right\}$$

$$\left. \begin{array}{l} -\ln p_1 - 1 = \lambda \\ -\ln p_2 - 1 = \lambda \\ \vdots \\ -\ln p_n - 1 = \lambda \\ \sum p_i = 1 \end{array} \right\} \Rightarrow \begin{array}{l} p_1 = p_2 = \dots = p_n \\ \text{and } p_i = \frac{1}{n} \quad i=1, \dots, n \\ \text{and } \lambda = -1 - \ln \frac{1}{n} = \ln n - 1. \end{array}$$

Hence the uniform distribution $p_1 = \frac{1}{n}, p_2 = \frac{1}{n}, \dots, p_n = \frac{1}{n}$ has maximum entropy.

Exercise 10 : Taken from [de Vries, Hillen et al. A Course in Math. Biol. p. 31, 32.

See p31-32.pdf