

Math 372 Assignment 1 Solutions

17C

Exercise 1 If there is no conveyor belt, then it doesn't matter where you tie your shoelaces and where you run. The conveyor belt gives you additional speed. And given your constraints, you want to maximize the benefit which you get from the belt. Hence you want to tie your shoe-laces on the belt, thereby extending the time you benefit from the belt. And you run off the belt as not to reduce the time on the belt too much.

Exercise (2)

Assume you bought a house and you accepted a loan of \$250.000 for 25 years on an annual interest rate of 4.5%.

a) What is your monthly payment

$$d_{n+1} = d_n \left(1 + \frac{4.5}{1200}\right) - M_n$$

$$\Rightarrow d_n = \left(1 + \frac{4.5}{1200}\right)^n d_0 - M_n \frac{1 - \left(1 + \frac{4.5}{1200}\right)^n}{\frac{4.5}{1200}}$$

$$25 \text{ years} \stackrel{!}{=} n = 12 \cdot 25 = 300$$

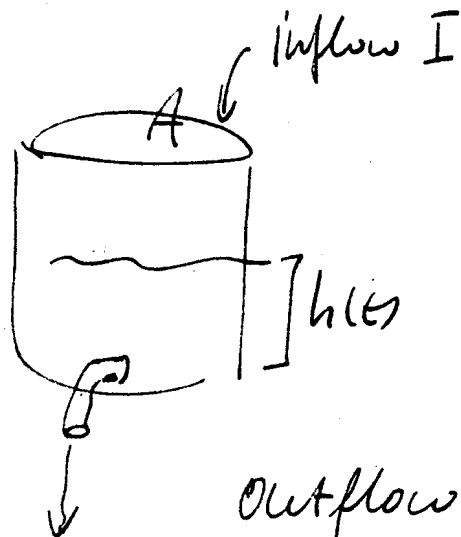
$$0 = d_{300} = \left(1 + \frac{4.5}{1200}\right)^{300} 250.000 - M_n \frac{1 - \left(1 + \frac{4.5}{1200}\right)^{300}}{-\frac{4.5}{1200}}$$

$$\begin{aligned} \Rightarrow M &= \frac{\left(1 + \frac{4.5}{1200}\right)^{300} 250.000 \cdot \frac{4.5}{1200}}{1 - \left(1 + \frac{4.5}{1200}\right)^{300}} \\ &= \frac{3.07 \cdot 250.000 \cdot \frac{4.5}{1200}}{1 - 3.07} \\ &= \$ 1392 \end{aligned}$$

Sensitivity on interest rate: $\%$

Part 2 see MAPLE file

Exercise 3



$$\text{outflow rate} = k\sqrt{h}$$

1. two ways: either volume

$$A \frac{dh}{dt} = I_v - k\sqrt{h}$$

or height:

$$\frac{dh}{dt} = I_h - k_h \sqrt{h}$$

which gives the same equation

$$\frac{dh}{dt} = a - b\sqrt{h}$$

$$2. \quad a - b\sqrt{h} = 0 \Rightarrow h^* = \frac{a^2}{b^2}$$

The steady state level h^* is larger for larger inflow rate a and smaller for larger outflow rate b .
Makes sense.

$$3. \quad f(h) = a - b\sqrt{h} \quad f'(h) = \frac{-b}{2\sqrt{h}} \Rightarrow h^* \text{ is stable for } h^* > 0.$$

If $h^* \rightarrow 0$ then $f'(h^*) \rightarrow -\infty$ which we could call unstable.

Exercise (4)

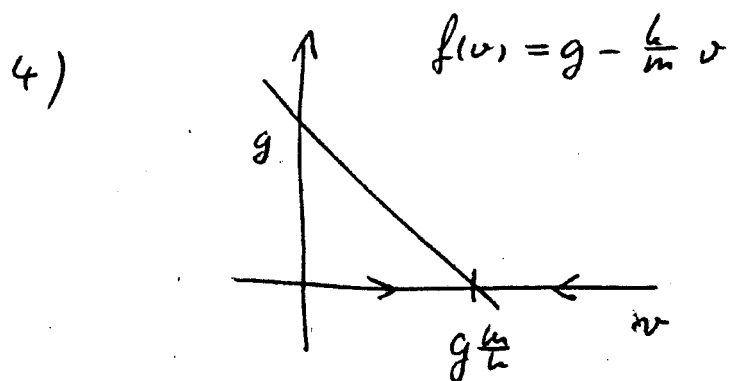
1) $ma = +mg - kv$

$$\frac{dv}{dt} = g - \frac{k}{m} v \quad v(0) = 0$$

2)
$$v(t) = v(0) e^{-\frac{k}{m} t} + \int_0^t e^{-\frac{k}{m} (t-s)} g \, ds$$

$$= g \frac{m}{k} (1 - e^{-\frac{k}{m} t})$$

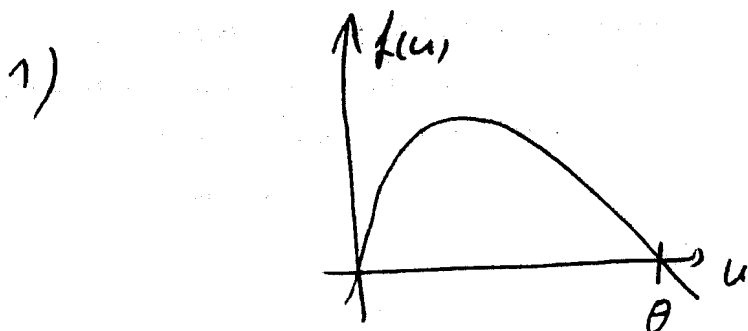
3) $\lim_{t \rightarrow \infty} v(t) = g \frac{m}{k}$



The steady state $v^* = g \frac{m}{k}$ is stable.
At terminal velocity, the acceleration due to gravity equals the deceleration due to air resistance.

Exercise 5

$$\dot{u} = -ru \ln\left(\frac{u}{\theta}\right) = f(u)$$



- 2) r : growth rate
 θ : carrying capacity

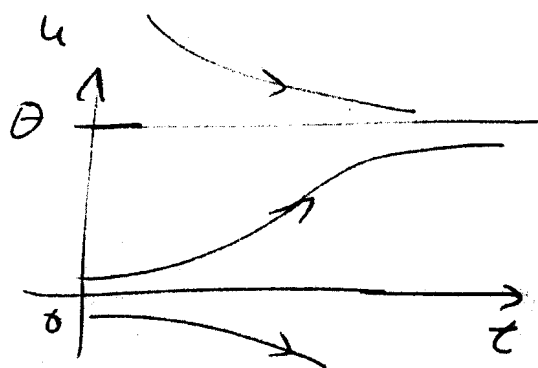
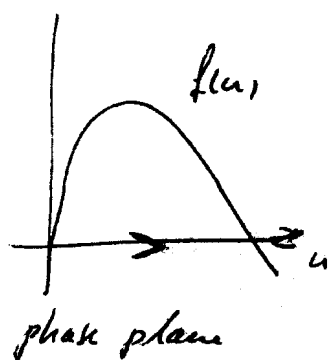
3) $f(u) = 0 \Leftrightarrow u_1^* = 0 \text{ or } u_2^* = \theta$

$$\begin{aligned} f'(u) &= -r \ln\left(\frac{u}{\theta}\right) - ru \frac{\theta}{u} \\ &= -r \ln \frac{u}{\theta} - r\theta \end{aligned}$$

$$f'(u_1^*) = f'(0) = +\infty \quad \text{super unstable}$$

$$f'(u_2^*) = f'(\theta) = -r\theta \quad \text{stable}$$

4)



$$\frac{\frac{du}{dt}}{u \ln\left(\frac{u}{\theta}\right)} = -r$$

$$\frac{\frac{du}{dt}}{u} = \frac{d}{dt} \ln u$$

$$\frac{d}{dt} \ln\left(\frac{u}{\theta}\right) = \frac{\theta}{u} \cdot \frac{du}{dt}$$

$$\frac{\theta \frac{du}{dt}}{u \ln\left(\frac{u}{\theta}\right)} = -r \theta$$

$$\frac{\frac{d}{dt} \ln\left(\frac{u}{\theta}\right)}{\ln\left(\frac{u}{\theta}\right)} = -r \theta$$

$$\frac{d}{du} \ln\left(\ln\left(\frac{u}{\theta}\right)\right) = -r \theta$$

integrate

$$\ln\left(\ln\frac{u(t)}{\theta}\right) - \underbrace{\left(\ln\ln\left(\frac{u(0)}{\theta}\right)\right)}_{C_1} = -r \theta t$$

$$\ln\left(\ln\frac{u}{\theta}\right) = C - r \theta t$$

$$\ln \frac{u}{\theta} = e^{C_1 - r\theta t} = C_2 e^{-r\theta t}$$

$$\frac{u}{\theta} = e^{C_2} e^{-r\theta t}$$

$$u = \theta e^{C_2} e^{-r\theta t}$$

$$C_2 = e^{C_1} = e^{\ln \ln \frac{u(0)}{\theta}} = \ln \frac{u(0)}{\theta}$$

$$u = \theta e^{\ln \frac{u(0)}{\theta}} e^{-r\theta t}$$

$$= \theta \left(\frac{u(0)}{\theta} \right) e^{-r\theta t}$$