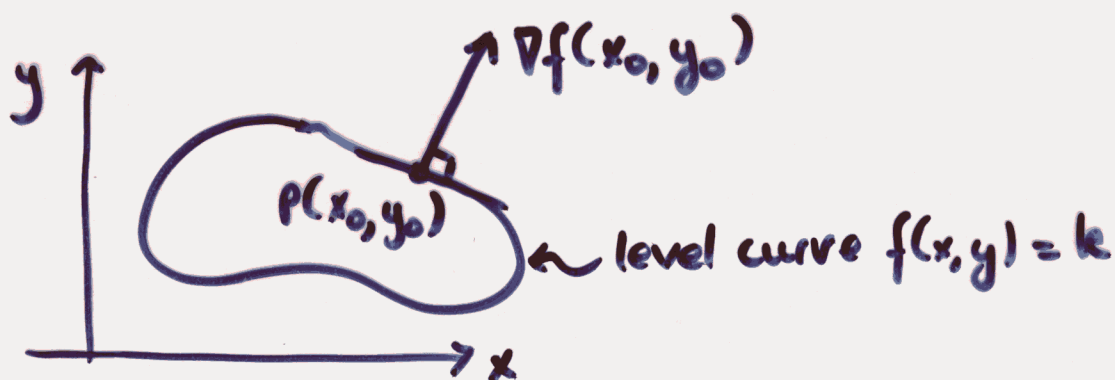
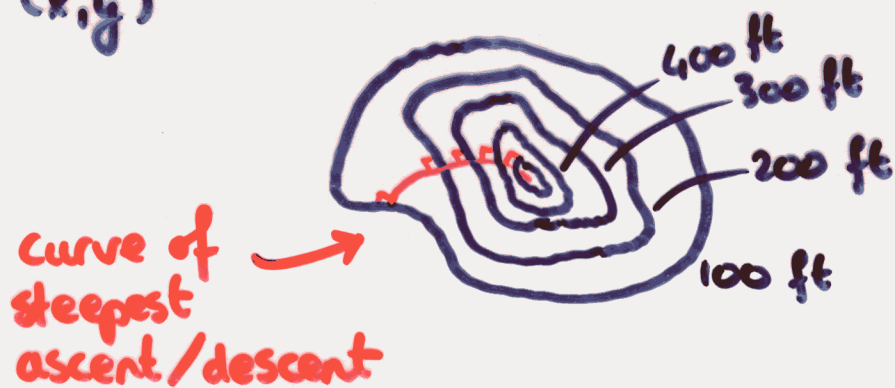


Recall (in 2 dimensions...)

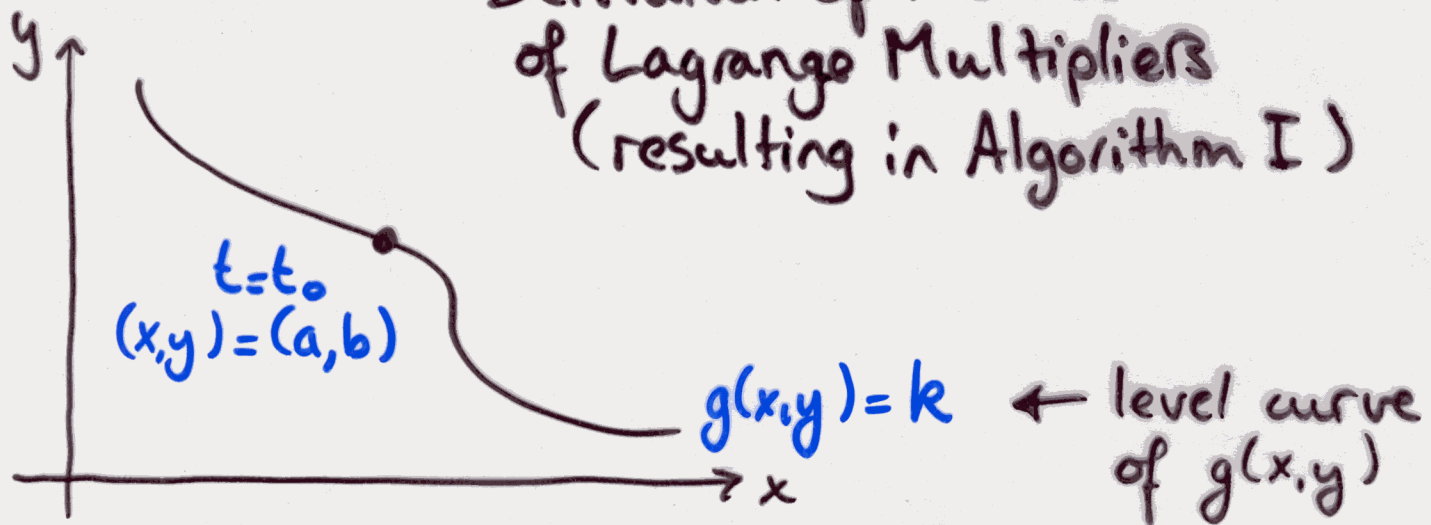
- The gradient vector for a function $f(x,y)$ is given by $\nabla f(x,y) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right) = (f_x, f_y)$
- Significance of the gradient vector
 - 1) $\nabla f(x_0, y_0)$ gives the direction of fastest increase of f
 - 2) $\nabla f(x_0, y_0)$ is orthogonal to the level curve of f through (x_0, y_0)



Eg: topographical map of a hill, with $f(x,y)$ = height above sea level at a point with coordinates (x,y)



Derivation of Method of Lagrange Multipliers (resulting in Algorithm I)



Let curve be parameterized by t , that is
 $x = x(t), \quad y = y(t)$

Consider $f(x,y) = f(x(t), y(t))$ along curve.

Suppose max (or min) of f on curve
 occurs at $t=t_0$, or $(x,y)=(a,b)$,
 with $a = x(t_0), \quad b = y(t_0)$

Must have $\frac{df}{dt}\bigg|_{t=t_0} = f_x(a,b) x'(t_0) + f_y(a,b) y'(t_0)$
 $= \nabla f(a,b) \cdot \underbrace{(x'(t_0), y'(t_0))}_{\text{tangent vector to curve at } (a,b)}$
 $= 0$

So $\nabla f(a,b)$ is $\perp$ to $g(x,y)=k$ at (a,b)

But we know that $\nabla g(a,b) \perp g(x,y)$ at (a,b)
 as well.

So !!! $\nabla f(a,b)$ and $\nabla g(a,b)$ are parallel

$$\boxed{\nabla f(a,b) = \lambda \nabla g(a,b)} \text{ for some } \lambda$$

Algorithm I

to find the absolute max or min of $f(x,y)$
on the curve $g(x,y) = k$

Evaluate $f(x,y)$ at all points (a,b) such
that

- ① $\nabla f(a,b) = \lambda \nabla g(a,b)$
AND $g(a,b) = k$
- ② $\nabla g(a,b) = \vec{0}$ AND $g(a,b) = k$
- ③ (a,b) is an endpoint of the curve
 $g(x,y) = k$

The absolute max and min of $f(x,y)$
are the largest and smallest values
obtained.

Comment

- ① gives 3 equations for 3 unknowns: a, b, λ
- ② gives 2 equations for 2 unknowns: a, b

Algorithm II

to find the absolute max or min of $f(x, y, z)$
on the surface $g(x, y, z) = k$

Evaluate $f(x, y, z)$ at all points (a, b, c)
such that

① $\nabla f(a, b, c) = \lambda \nabla g(a, b, c)$

AND $g(a, b, c) = k$

② $\nabla g(a, b, c) = \vec{0}$ AND $g(a, b, c) = k$

③ (a, b, c) is a boundary point on the
surface $g(x, y, z) = k$

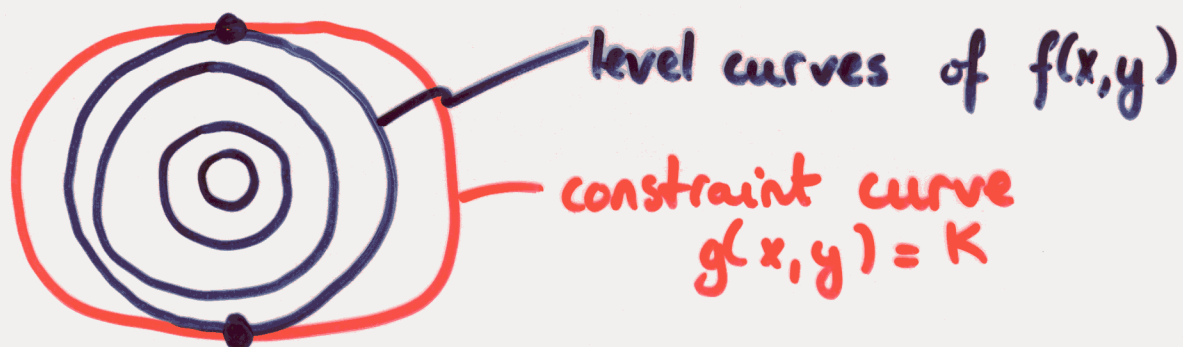
The absolute max and min of $f(x, y, z)$
are the largest and smallest values
obtained

Comment

① gives 4 equations for 4 unknowns: a, b, c, λ

② gives 3 equations for 3 unknowns: a, b, c

Geometric interpretation for algorithm I:



In general, absolute extrema of $f(x, y)$ on $g(x, y) = K$ occur at points where the level curves meet tangentially

Geometric interpretation for algorithm II:

Now, absolute extrema of $f(x, y, z)$ on $g(x, y, z) = K$ occur at points where level surfaces meet tangentially

Algorithm III

to find the absolute max or min of $f(x, y, z)$
subject to the two constraints

$$g(x, y, z) = K \quad \text{and} \quad h(x, y, z) = L$$

Evaluate $f(x, y, z)$ at all points (a, b, c)
such that

looking for
extrema of f
on intersection
of $g=K$ and
 $h=L$

$$\textcircled{1} \quad \nabla f(a, b, c) = \lambda \nabla g(a, b, c) + \mu \nabla h(a, b, c)$$

$$\text{AND} \quad g(a, b, c) = K$$

$$\text{AND} \quad h(a, b, c) = L$$

Obtain 5 equations
for 5 unknowns:
 a, b, c, λ, μ

$$\textcircled{2} \quad \nabla g(a, b, c) \text{ and } \nabla h(a, b, c) \text{ are} \\ \text{linearly dependent}$$

$$\text{AND} \quad g(a, b, c) = K$$

$$\text{AND} \quad h(a, b, c) = L$$

$$\textcircled{3} \quad (a, b, c) \text{ is an endpoint of the curve} \\ \text{defined by} \\ g(x, y, z) = K \text{ and } h(x, y, z) = L$$

The absolute max and min of $f(x, y, z)$
are the largest and smallest values
obtained.

Geometric interpretation for algorithm III:

The equation

$$\nabla f = \lambda \nabla g + \mu \nabla h$$

says that ∇f must lie in the plane that is spanned by ∇g and ∇h

(provided ∇g and ∇h are linearly independent)